Adjoints and Importance in Rendering: An Overview

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Abstract—This survey gives an overview of the use of importance, an adjoint of light, in speeding up rendering. The importance of a light distribution indicates its contribution to the region of most interest—typically the directly visible parts of a scene. Importance can therefore be used to concentrate global illumination and ray tracing calculations where they matter most for image accuracy, while reducing computations in areas of the scene that do not significantly influence the image. In this paper, we attempt to clarify the various uses of adjoints and importance in rendering by unifying them into a single framework. While doing so, we also generalize some theoretical results—known from discrete representations—to a continuous domain.

Index Terms—Rendering, adjoints, importance, light, ray tracing, global illumination, participating media, literature survey.

1 Introduction

The use of importance functions started in neutron transport simulations soon after World War II. Importance was used (in different disguises) from 1983 to accelerate ray tracing [2], [16], [27], [78], [79]. Smits et al. [67] formally introduced the use of importance for global illumination in 1992. Since then, importance has been used to optimize finite element methods (radiosity [7], [20] and radiance [4], [12]) and various Monte Carlo methods (path tracing [29], [42], [73], random walk radiosity [56], stochastic relaxation radiosity [49], ray bundles [72], and photon particle tracing for photon maps [38], [58], [71]). Importance also provides the theoretical foundation for algorithms such as bidirectional path tracing [40], [41], [76]. To date, there have been more than 60 publications describing various uses of importance to increase rendering efficiency. This multitude of publications can be overwhelming and confusing. Part of the confusion stems from the fact that there are six commonly used representations of light (incident and exitant radiance, radiosity, irradiance, and incident and exitant power), six corresponding representations of importance (incident and exitant directional importance, incident and exitant diffuse importance, and incident and exitant power importance), and several different inner products used to define adjoints. Also, some methods use a continuous framework while others use a discrete approximation and different authors use different notation and terminology. This paper is an attempt to clarify and categorize the uses of importance in rendering so far.

Importance is defined as a specific adjoint. In general, an integral equation has infinitely many adjoint equations, each with a different source term. If the source term is given, the adjoint equation and its solution are unique. The solution to the adjoint equation with a source term at the most important part of the function domain is called *importance* since it indicates how much the different parts of the domain contribute to the solution at the most important part. Importance is also known as *visual importance*, *view importance*, *potential*, *visual potential*, *value*, or *potential value*.

For rendering, the integral equations we are concerned with express light transport and importance is defined as the adjoint of light that has a source term at the region of most interest; typically, this region is the eye point, the image plane, or the directly visible parts of the scene. Importance expresses the fraction of light that makes it to the region of interest. It turns out that importance is transported like light. There is an intuitive illustration of this: If we turn the light sources off, shine light from the important region (for example, from the image plane in the directions within the field of view), and let that light bounce in the scene until it reaches equilibrium, then the contribution (to the image) of different parts of the scene is proportional to the amount of light reaching those parts. Importance is very useful for rendering since it enables us to focus the computations on the light that contributes most to the image.

The rest of this paper is organized as follows: We begin with an example of how importance improves the efficiency of a global illumination solution method. We then provide an overview of the necessary mathematical formalism: inner products and adjoints. Next, we describe various representations of light and importance, followed by a discussion of the adjoint relationships between these representations. We also provide physical interpretations and discuss importance source terms. A comprehensive overview of publications about importance in rendering, as well as the most pertinent publications from neutron transport theory, follows. A conclusion and discussion of future work is given at the end. It is assumed that the reader is familiar with standard global illumination terms such as radiance, radiosity, power, bidirectional scattering distribution function, and geometric term. (If not, several textbooks provide excellent introductions [15], [25], [64].) No prior knowledge of adjoints or importance is assumed.

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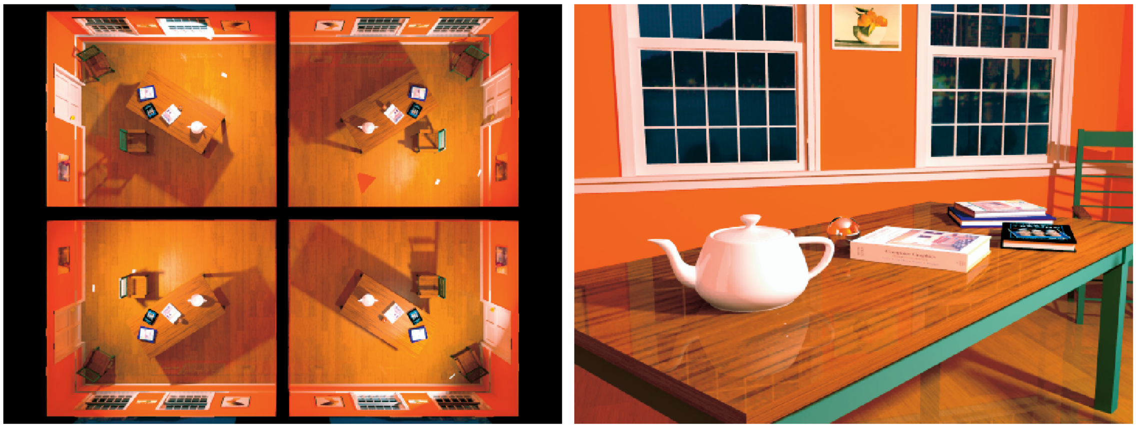
Manuscript received 1 Aug. 2001; revised 5 Feb. 2002; accepted 16 May 2002. For information on obtaining reprints of this article, please send e-mail to: tvcg@computer.org, and reference IEEECS Log Number 114631.



(a) (b)

Fig. 1. Orange interior: (a) entire scene seen from above; (b) seen from the intended viewpoint.

2 Example: Importance for Photon Tracing

This section provides a simple example of the use of importance in rendering. The purpose is to give an intuition about how importance is emitted and transported and how it is used to guide the accuracy of the light calculation. In this example, we use the photon map method [30], [31], but importance is just as applicable to many other global illumination methods.

Fig. 1a shows our example scene, four rooms with closed doors between them. Fig. 1b is the image we are interested in computing, a close-up of the tabletop in the upper right room. The red pyramid in Fig. 1a indicates the view point and directions for Fig. 1b.

First, importance particles ("importons") [38], [58], [71] are emitted from the intended viewpoint in directions within the viewing frustum. These particles are traced around the scene and stored every time they hit a diffuse surface. The stored particles are shown in Fig. 2a. (The emitted importance particles are white, but they change color at each bounce according to the color of the surfaces they hit. In this scene, most particles turn orange.) After the importance particle tracing, the importance is estimated at each importance particle location using the local density of importance particles. These importance estimates are shown in Fig. 2b. Note that no importance reaches adjacent rooms since the doors are closed; this reflects the fact that no light from adjacent rooms reaches the view point.

Then, the photon tracing phase follows. We use the importance estimates to determine photon storage probabilities. At locations with low importance, we use Russian roulette to decide whether to store the photon or not; if the photon is stored, its power is increased to compensate for the low storage probability. Compare the top row of Fig. 3a with the bottom row of Fig. 3: Importance was not used in the top row of Fig. 3, so most photons are stored in bright regions, no matter how unimportant those regions are. In contrast, the bottom row of Fig. 3 shows the gain from using importance—most photons are stored in areas that are either directly visible from the intended viewpoint or are significantly influencing the illumination there. Due to the higher concentration of photons, the illumination is approximated much more accurately in Fig. 3f than in Fig. 3c. The most visible difference is the sharper shadows.

(a) (b)

Fig. 2. Importance in interior scene, seen from above: (a) 100,000 importance particles; (b) importance estimates.

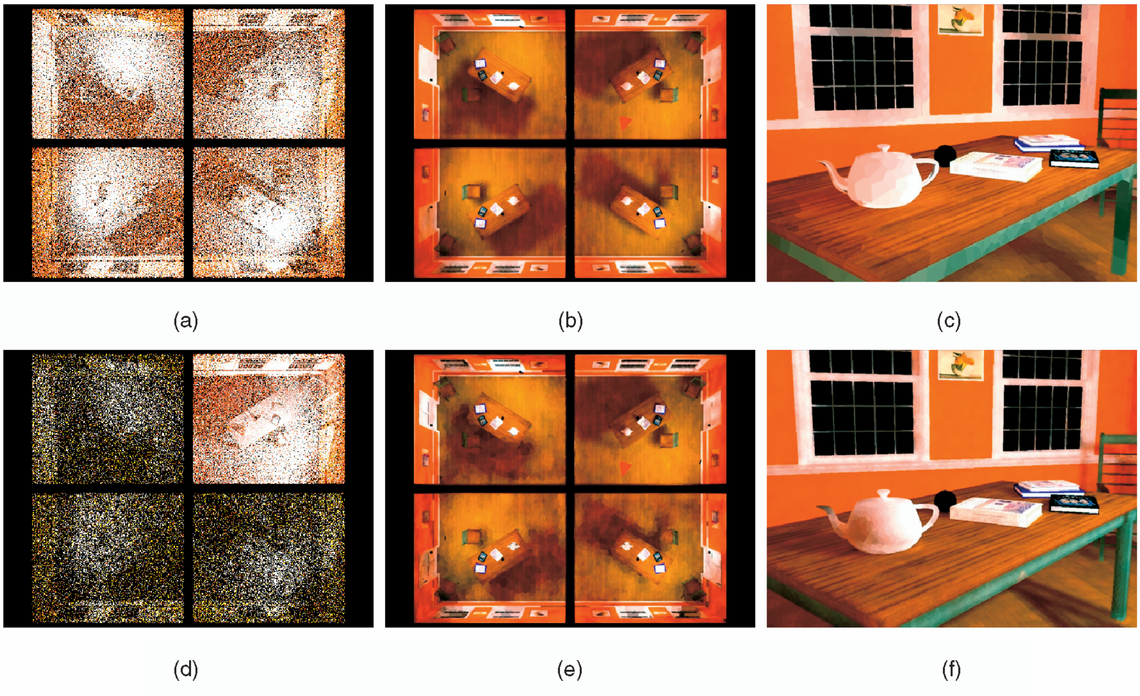


Fig. 3. Light in interior scene: (a) 500,000 photons stored without importance; (b)-(c) radiance estimates based on the photons in (a); (d) 500,000 photons stored using importance; (e)-(f) radiance estimates based on the photons in (d).

In this example, importance is only used to determine photon storage. It is also possible to use importance to guide photon emission and reflection [31], [58], but that is beyond the scope of this example.

3 Mathematical Background

Light and importance are adjoint functions. But what exactly does that mean? This section introduces mathematical concepts—inner products and adjoints—that enable us to define the precise relationships between light and importance.

3.1 Inner Products

The *inner product* of two functions, f and g, defined on domain D with measure function μ, is

$$\langle f \mid g \rangle_\mu \equiv \int_D f(u)\, g(u)\, d\mu(u).$$

Among other uses in rendering, we use inner products to compute scalar functions of radiance distributions. For example, assume that we know the (exitant) radiance L_o everywhere in a scene and are interested in the average radiance through a single pixel of the image plane. We can compute this average by integrating the radiance with a weighting function W that is nonzero only in the pixels' part of the image plane. Written as an inner product, the integral is $\langle L_o \mid W \rangle_\mu$.

We use inner products with different measure functions for functions defined on different domains. Consider first the inner product of an exitant and an incident directional quantity, i.e., a quantity leaving some surfaces and another quantity impinging on the same surfaces—for example, exitant radiance and incident directional importance. In this case, the integration is over points x on all surfaces S and all (exitant) directions ω on the hemisphere Ω above each point. The measure function μ is the solid angle times projected area: $\cos\theta_x\, d\omega\, dA_x$. The inner product is then

$$\langle f \mid g \rangle_{\omega A'} \equiv \int_S \int_{\Omega_o} f_o(x,\omega)\, g_i(-\omega, x)\, \cos\theta_x\, d\omega\, dA_x.$$

Another useful inner product is between two exitant directional quantities (at different locations). Here, the domain of integration is all pairs of points in the scene. The measure function μ is the geometric term G (including a visibility term) times the differential areas at the two points; this measures "beam throughput" [15]. The inner product is then

$$\langle f \mid g \rangle_{GAA} \equiv \int_S \int_S f_o(x, \omega_{xy})\, g_o(y, \omega_{yx})\, G(x,y)\, dA_x\, dA_y.$$

For diffuse and power quantities, we use simpler integration measures; these will be introduced in Section 5.

3.2 Adjoint Operators

An *operator* is a transformation of one function to another function. Examples are integration, differentiation, and functional inversion. In global illumination simulation, a commonly used operator is the "exitant transport operator" $\mathcal{T}_o$ that expresses one bounce of exitant light: the exitant radiance distribution that results from one bounce of some other exitant radiance distribution. Another common operator is the "incident transport operator" $\mathcal{T}_i$ that similarly expresses one bounce of incident radiance. Two operators $\mathcal{O}$ and $\mathcal{O}^*$ are *adjoint* (with respect to an inner product with measure μ) if

$$\langle \mathcal{O}f \mid g\rangle_\mu = \langle f \mid \mathcal{O}^* g\rangle_\mu$$

for all functions f and g. It will be shown in Section 5 that $\mathcal{T}_o$ and $\mathcal{T}_i$ are adjoint operators.

The adjoint of an operator is unique (except for a set of measure zero) [45]. An operator is *self-adjoint* if $\mathcal{O}^* = \mathcal{O}$. Using the definition of adjoint operators, it is easy to verify the following three algebraic rules: The adjoint of the adjoint of an operator is the operator itself, $(\mathcal{O}^*)^* = \mathcal{O}$. The adjoint operator is linear, $(\mathcal{O} + \mathcal{P})^* = \mathcal{O}^* + \mathcal{P}^*$. The adjoint of an operator composed of two operators is the composition of the adjoint operators in reverse order, $(\mathcal{O}\mathcal{P})^* = \mathcal{P}^*\mathcal{O}^*$.

3.3 Adjoint Equations and Functions

Given an equation $\mathcal{O}f = f'$, its *adjoint equations* are the equations $\mathcal{O}^* g = g'$ for all functions g'. (The functions f' and g' are usually called "source terms" in computer graphics, but the terms "boundary conditions" and "driving terms" are common in other fields.) Two equations are adjoint if they involve adjoint operators. For example, the equilibrium equation for exitant radiance ($L_o = L_{o,e} + \mathcal{T}_o L_o$) and the equilibrium equation for incident radiance ($L_i = L_{i,e} + \mathcal{T}_i L_i$) are adjoint equations. This will be discussed in detail in Section 5.

Two functions f and g are *adjoint functions* if they satisfy adjoint equations. From the previous example, we see that exitant radiance L_o and incident radiance L_i are adjoint functions.

4 Light and Importance

In order to define the precise relationships between light and importance, we need to first examine the different representations of light and importance and the operators used to transport them.

4.1 Light

There are six commonly used representations of light: exitant and incident radiance, radiosity, irradiance, and exitant and incident power.

The canonical representation of light is the exitant radiance $L_o(x, \omega)$ leaving point x in direction ω. Incident radiance $L_i(\omega, y)$ is the light reaching point y from direction ω. (Incident radiance is also sometimes called "field radiance" [1].) Radiance is constant along an unobstructed ray in a nonparticipating medium, so the relationship between incident and exitant radiance is $L_i(\omega_{xy}, y) = L_o(x, \omega_{xy})$ when points x and y are mutually visible. (Here, ω_{xy} is the direction from x to y.)

Radiosity $B(x)$ at a point x is the cosine-weighted integral of the exitant radiance over the exitant hemisphere at x: $B(x) = \int_\Omega L_o(x, \omega) \cos\theta_x \, d\omega$. Radiosity is therefore independent of direction and is sufficient to characterize the light reflected from a diffuse surface. Similarly, irradiance $E(y)$ is the cosine-weighted integral of incident radiance over the incident hemisphere at point y. (For consistent notation, we could use the symbol B_i for irradiance, but E is standard.)

The exitant power dP_o at a point is the product of the radiosity there and the area of an infinitesimal region around the point: $dP_o(x) = B(x)\, dA_x$. Like radiosity, exitant power is the same in all directions on the hemisphere above the point. The incident power dP_i at a point is the product of irradiance and the area of an infinitesimal region around the point: $dP_i(x) = E(x)\, dA_x$.

TABLE 1
(a) Light Representations and Units;
(b) Importance Representations

Light representation	Units
exitant radiance L_o	[W/m^{2}sr]
incident radiance L_i	[W/m^{2}sr]
radiosity B	[W/m^2]
irradiance E	[W/m^2]
exitant power dP_o	[W]
incident power dP_i	[W]

(a)

Importance representation
exit. directional imp. W_o
inc. directional imp. W_i
exit. diffuse imp. I_o
inc. diffuse imp. I_i
exit. power imp. dJ_o
inc. power imp. dJ_i

(b)

These six different representations of light are listed in Table 1a. More details on light representations and light transport can be found in, e.g., the textbook by Cohen and Wallace [15].

4.2 Importance

Several definitions of importance have been used in the literature; the definition determines which units should be assigned to importance. Fortunately, the exact definition does not influence the equilibrium equations. So, we will use one definition in the following discussion and return to the topic of the exact units of importance in Section 5.5.

The importance of radiance $L_o(x, \omega)$ or $L_i(\omega, y)$ is commonly defined as the fraction of that radiance that contributes (directly or indirectly) to the region of interest. Consequently, if radiance is directly incident onto the region of interest, its importance is 1.

Importance is often represented as the exitant directional importance $W_o(x, \omega)$ from a point x in direction ω or the incident directional importance $W_i(\omega, y)$ from a direction ω to a point y. What is the relationship between incident and exitant directional importance? Since radiance is constant along an unobstructed ray, the contribution of that radiance —and therefore the importance—must also be constant along that ray. Hence, the relationship between incident and exitant directional importance is $W_i(\omega_{xy}, y) = W_o(x, \omega_{xy})$ when x and y are mutually visible.

We can also define diffuse importance quantities similar to radiosity and irradiance: Exitant diffuse importance I_o is the cosine-weighted integral of the exitant directional importance over the exitant hemisphere at x, $I_o(x) = \int_\Omega W_o(x, \omega) \cos\theta_x \, d\omega$. Similarly, incident diffuse importance $I_i(y)$ is the cosine-weighted integral of incident directional importance over the incident hemisphere at point y.

Finally, we can define power-like importance quantities: Exitant power importance J_o at a point is the product of the exitant diffuse importance there and an infinitesimal area around the point, $dJ_o(x) = I_o(x)\,dA_x$. The incident power importance dJ_i at a point is the product of incident diffuse importance and an infinitesimal area around the point: $dJ_i(x) = I_i(x)\,dA_x$.

These six representations of importance are listed in Table 1b.

Note that (visual) importance is different from the importance function used in *importance sampling*. Importance sampling is a general Monte Carlo method which increases local sampling efficiency by concentrating samples in places where the contribution to the total value is expected to be high. (This is also known as "sample density biasing.") Visual importance is optimal for importance sampling, but, in general, any function that might reduce variance can be used. General importance sampling is outside the scope of this paper, but detailed descriptions can be found in [25], [36]. (There is also a nice paper by Veach and Guibas [77] that describes how to importance sample using a combination of multiple importance functions.)

4.3 Operators

Operator notation is very convenient to describe transport of light and importance. We most often use the four operators $\mathcal{P}$, $\mathcal{S}$, $\mathcal{T}_o$, and $\mathcal{T}_i$. The propagation operator $\mathcal{P}$ converts exitant radiance to incident radiance: $L_i = \mathcal{P}L_o$ (recall that $L_i(\omega_{xy}, y) = L_o(x, \omega_{xy})$ if x and y are mutually visible) and exitant directional importance to incident directional importance: $W_i = \mathcal{P}W_o$. The scattering operator $\mathcal{S}$ converts incident radiance to exitant radiance by reflection or transmission: $L_o(y,\omega) = \int f_s(\omega', y, \omega)\, L_i(\omega', y)\, \cos\theta'_y\, d\omega'$ or, more concisely, $L_o = \mathcal{S}L_i$. If the BSDF f_s is symmetric, the scattering operator also converts incident directional importance to exitant directional importance, $W_o = \mathcal{S}W_i$. (We can only use the same BSDF f_s for importance as for light if f_s is symmetric [74].) The *exitant transport operator* is the composition of propagation and scattering, $\mathcal{T}_o = \mathcal{SP}$. The *incident transport operator* is the composition of scattering and propagation, $\mathcal{T}_i = \mathcal{PS}$. Both transport operators express one bounce of light, as shown in Fig. 4.

The operators are simpler for light transport between diffuse surfaces. Radiosity is transported by the exitant diffuse transport operator $\mathcal{T}_{o,d}$ and irradiance is transported by the incident diffuse transport operator $\mathcal{T}_{i,d}$. Exitant power is transported by the exitant power transport operator $\mathcal{T}_{o,p}$ and incident power is transported by the incident power transport operator $\mathcal{T}_{i,p}$. The definitions of these operators are listed in Table 2. With these operators, it is simple to write equilibrium equations for each type of light and importance—see Table 3.

5 Adjoints in Rendering

Given the formal definitions of adjoints in Section 3 and the description of the different representations of light and importance in Section 4, we are now ready to look at the precise adjoint relationships between light and importance.

5.1 Directional Operators and Functions

Writing out the inner products and manipulating integrals, we can show that $\langle \mathcal{P}f_o \mid g_o\rangle_{\omega A'} = \langle f_o \mid \mathcal{P}g_o\rangle_{\omega A'}$. Therefore, the propagation operator $\mathcal{P}$ is, by definition, self-adjoint, $\mathcal{P}^* = \mathcal{P}$. Similarly, we can also show that the scattering operator $\mathcal{S}$ is self-adjoint, $\mathcal{S}^* = \mathcal{S}$, if the BSDF is symmetric. (Most real BSDFs are symmetric, the most notable exception being refraction [74].)

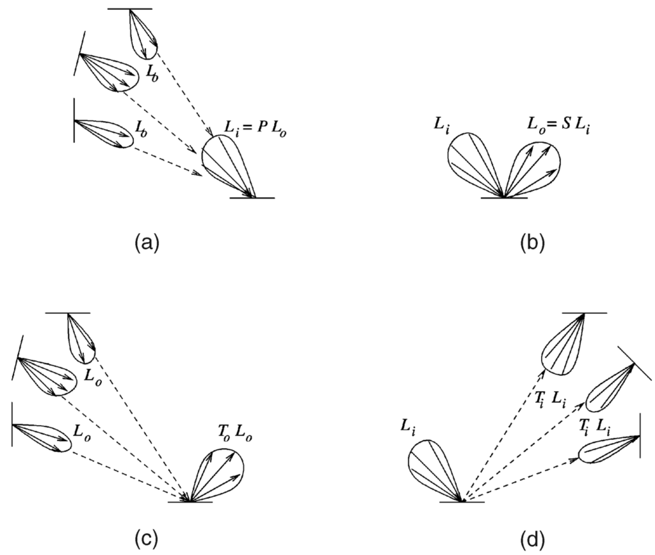


Fig. 4. Operators: (a) propagation operator $\mathcal{P}$; (b) scattering operator $\mathcal{S}$; (c) exitant transport operator $\mathcal{T}_o = \mathcal{SP}$; (d) incident transport operator $\mathcal{T}_i = \mathcal{PS}$.

Since $\mathcal{P}$ and $\mathcal{S}$ are both self-adjoint, it can be seen directly from their definitions that $\mathcal{T}_o$ and $\mathcal{T}_i$ are adjoint: $\mathcal{T}_o^* = (\mathcal{SP})^* = \mathcal{P}^*\mathcal{S}^* = \mathcal{PS} = \mathcal{T}_i$. Hence,

$$\langle \mathcal{T}_o f_o \mid g_i\rangle_{\omega A'} = \langle f_o \mid \mathcal{T}_i g_i\rangle_{\omega A'}.$$

It follows directly that incident radiance L_i and incident directional importance W_i are adjoint functions of exitant radiance L_o and exitant directional importance W_o. This important result has been shown by numerous authors [1], [3], [4], [5], [19], [20], [21], [37], [39], [41], [51], [63], [74], [75], [76].

It is most common to use L_o and W_i for global illumination calculations. However, it is more practical for some algorithms to only propagate exitant quantities, i.e., L_o and W_o. Then, the inner product can be expressed as $\langle L_o \mid \mathcal{P}W_o\rangle_{\omega A'}$ or $\langle \mathcal{P}L_o \mid W_o\rangle_{\omega A'}$. It is interesting to note that the value of this inner product can also be expressed as an inner product with the "beam throughput" measure: $\langle L_o \mid W_o\rangle_{GAA}$.

(An aside: Using an inner product with two areas in the integration measure, it can be shown that Kajiya's two-point transport intensity [34] (which has units $[\mathrm{W/m^4}]$) and a similarly defined two-point importance are adjoint functions of exitant radiance L_o and exitant importance W_o. This was shown by Christensen et al. [11] (although two-point importance was confusingly called incident importance there) and by Arvo [1]. It can also be shown that incident radiance L_i and incident directional importance W_i are adjoint functions of two-point transport intensity and two-point importance.)

TABLE 2
Transport Operators

Operator	Operator definition
$\mathcal{T}_o = \mathcal{SP}$	$(\mathcal{T}_o L_o)(y,\omega) \equiv \int_{\Omega_i} f_s(\omega', y, \omega)\, L_o(x,\omega')\, \cos\theta'_y\, d\omega'$
$\mathcal{T}_i = \mathcal{PS}$	$(\mathcal{T}_i L_i)(\omega, y) \equiv \int_{\Omega_i} f_s(\omega', x, \omega)\, L_i(\omega', x)\, \cos\theta'_x\, d\omega'$
$\mathcal{T}_{o,d}$	$(\mathcal{T}_{o,d} B)(y) \equiv f_s(y) \int_{\Omega_i} B(x)\, \cos\theta_y\, d\omega$
$\mathcal{T}_{i,d}$	$(\mathcal{T}_{i,d} E)(y) \equiv \int_{\Omega_i} f_s(x)\, E(x) \cos\theta_y\, d\omega$
$\mathcal{T}_{o,p}$	$(\mathcal{T}_{o,p} dP_o)(y) \equiv dA_y\, f_s(y) \int_S G(x,y)\, dP_o(x)$
$\mathcal{T}_{i,p}$	$(\mathcal{T}_{i,p} dP_i)(y) \equiv dA_y \int_S f_s(x)\, G(x,y)\, dP_i(x)$

The notation $(\mathcal{T}_o L_o)(y,\omega)$ denotes the result of $\mathcal{T}_o$ operating on $L_o(x,\omega')$ to produce a function whose argument is (y,ω).

5.2 Diffuse Operators and Functions

For purely diffuse reflection, it can be shown that

$$\langle \mathcal{T}_{o,d} f_o \mid g_i \rangle_{\omega A'} = \langle f_o \mid \mathcal{T}_{i,d} g_i \rangle_{\omega A'}.$$

Therefore, $\mathcal{T}_{o,d}$ and $\mathcal{T}_{i,d}$ are adjoint operators and irradiance E and diffuse incident importance I_i are adjoint functions of radiosity B and exitant diffuse importance I_o. This was informally shown for the discrete case by Pattanaik and Mudur [55]. (We can derive the same adjoint relationships using inner products with measures $d\mu = dA_x\, dA_y$ and $d\mu = dA_x$.)

5.3 Mixed Diffuse and Power Operators and Functions

Still considering purely diffuse reflection, one can show that:

$$\langle \mathcal{T}_{o,d} f_o \mid dg_i \rangle = \langle f \mid \mathcal{T}_{i,p} dg_i \rangle.$$

(f_o is a radiosity-like function, dg_i is an incident power-like function, and the inner product has no integration measure.) So, $\mathcal{T}_{o,d}$ and $\mathcal{T}_{i,p}$ are adjoint operators under this simple inner product and incident power dP_i and incident power importance dJ_i are adjoint functions of radiosity B and exitant diffuse importance I_o. Smits et al. [67], Sbert et al. [61], [62], and Bekaert [5] showed a discretized version of this. It can be shown in a similar manner that

$$\langle \mathcal{T}_{o,p} df_o \mid g_i \rangle = \langle df_o \mid \mathcal{T}_{i,d} g_i \rangle.$$

(Here, df_o is an exitant power-like function and g_i is an irradiance-like function.) So, $\mathcal{T}_{o,p}$ and $\mathcal{T}_{i,d}$ are adjoint operators, and irradiance E and incident diffuse importance I_i are adjoint functions of exitant power dP_o and exitant power importance dJ_o. This was used by Pattanaik and Mudur [51], [54], [56] and shown formally by Sbert et al. [61], [62] and Bekaert [5] for the discrete case.

5.4 Power Operators and Functions

Still considering purely diffuse reflection, it can be shown that

TABLE 3
Equilibrium Equations for Light and Importance

Light	Importance
$L_o = L_{o,e} + \mathcal{T}_o L_o$	$W_o = W_{o,e} + \mathcal{T}_o W_o$
$L_i = L_{i,e} + \mathcal{T}_i L_i$	$W_i = W_{i,e} + \mathcal{T}_i W_i$
$B = B_e + \mathcal{T}_{o,d} B$	$I_o = I_{o,e} + \mathcal{T}_{o,d} I_o$
$E = E_e + \mathcal{T}_{i,d} E$	$I_i = I_{i,e} + \mathcal{T}_{i,d} I_i$
$dP_o = dP_{o,e} + \mathcal{T}_{o,p} dP_o$	$dJ_o = dJ_{o,e} + \mathcal{T}_{o,p} dJ_o$
$dP_i = dP_{i,e} + \mathcal{T}_{i,p} dP_i$	$dJ_i = dJ_{i,e} + \mathcal{T}_{i,p} dJ_i$

$$\langle \mathcal{T}_{o,p} df_o \mid dg_i \rangle_{1/A} = \langle df_o \mid \mathcal{T}_{i,p} dg_i \rangle_{1/A}.$$

(Here, df_o and dg_i are both power-like functions, and the integration measure is $d\mu = 1/A_x$.) Therefore, $\mathcal{T}_{o,p}$ and $\mathcal{T}_{i,p}$ are adjoint operators and incident power dP_i and incident power importance dJ_i are adjoint functions of exitant power dP_o and exitant power importance dJ_o. Neumann et al. [49] and Prikryl et al. [60] used the fact that dP_o and dJ_i are adjoint.

These adjoint relationships are summarized in Table 4.

5.5 Physical Interpretation of Inner Products

There are three commonly used interpretations of the inner product of exitant radiance and the source term for incident directional importance, $\langle L_o \mid W_{i,e} \rangle_{\omega A'}$. One interpretation is that the inner product is the power that radiance distribution L_o contributes to the image. With this interpretation, the inner product has unit [W] and, therefore, directional importance must be dimensionless. Another interpretation is that the inner product is the (dimensionless) response of a measuring device at the viewpoint as result of radiance distribution L_o; with this interpretation, the unit of directional importance must be $[W^{-1}]$. Other authors define dJ_i to be a dimensionless fraction [67] and the unit of directional importance follows from that. These differences of interpretation are mostly academic; the usefulness of importance is independent of the units assigned to it. We have chosen the first interpretation in this paper.

5.6 Source Terms for Importance

In general, any part of the scene can be defined to be most important and, hence, be the source of importance. For example, a particular art piece in the middle of the image may be the center of attention and, hence, requires the most accurate solution. Very often, everything that is directly visible in the image is defined to be very important. In that case, the optimal importance source term depends on the error metric used to measure image quality.

If the error metric measures total image error, then the image plane should emit importance 1 in all directions within the viewing frustum. This results in the illumination at visible surfaces being computed to an accuracy proportional to their projected area in screen space. This was used by Smits et al.

TABLE 4
Adjoint Operators and Functions

f	g	O	O^*	measure μ
L_o, W_o	L_i, W_i	$\mathcal{T}_o$	$\mathcal{T}_i$	$\omega A'$
B, I_o	E, I_i	$\mathcal{T}_{o,d}$	$\mathcal{T}_{i,d}$	$\omega A'$, A
B, I_o	dP_i, dJ_i	$\mathcal{T}_{o,d}$	$\mathcal{T}_{i,p}$	—
dP_o, dJ_o	E, I_i	$\mathcal{T}_{o,p}$	$\mathcal{T}_{i,d}$	—
dP_o, dJ_o	dP_i, dJ_i	$\mathcal{T}_{o,p}$	$\mathcal{T}_{i,p}$	$1/A$

[67], Christensen et al. [11], Bekaert and Willems [7], Neumann et al. [49], and implicitly by Dutré et al. [20].

If the error metric measures maximum pixel error, then all visible patches should emit importance 1. Pattanaik and Mudur [54] used a slight variation on this by initially assigning importance 1 to all visible patches, but stopping importance emission from patches which achieve sufficient accuracy during calculations.

Final gathering complicates the issue somewhat. Final gathering is the use of one level of distribution ray tracing at directly visible diffuse surfaces. When final gathering is used for rendering the image, we are no longer interested in getting the most accurate global illumination simulation results at the directly visible patches. Instead, the most important parts of the scene are those that contribute light directly to the directly visible parts (i.e., one bounce away from the image plane). Hence, only indirect importance should be used to guide solution accuracy. (Neumann et al. [49] made a similar observation in a different setting: Only indirect importance should determine how many particles (rays) to shoot from a patch in stochastic relaxation radiosity. Bekaert [5] gave a proof of the lower variance with this choice.) For a more in-depth discussion of the subtleties of the use of importance in a final gathering setting, see Suykens and Willems [71].

6 Literature Survey

The following is a fairly complete description of articles, books, and dissertations related to the usage of adjoints and importance in rendering. The publications are divided into the following six categories: mathematics and nuclear physics, theoretical results in rendering, "classic" and distribution ray tracing, finite element global illumination, Monte Carlo global illumination, and participating media.

6.1 Background Material: Mathematics and Nuclear Physics

The term "adjoint equation" was first used by Lagrange and adjoint equations were used by Fredholm in 1903 to determine the solvability of certain integral equations [24]. The adjoint of neutron density was used early in the development of the Monte Carlo method to speed up simulations of neutron transport. According to Malvin Kalos [personal communication, 1999], it was von Neumann who first pointed out the significance of the adjoint function in variance reduction in Monte Carlo transport calculations. Discussions between Feshbach, Friedman, Goertzel, and Kahn at the Oak Ridge National Laboratory in the summer of 1949 led to the insight that the optimal importance sampling function is equivalent to the solution of an adjoint problem [32]. The first papers describing this relationship were published by Goertzel [26] and Kahn and Harris [32], [33] later in 1949. The term "importance function" was coined by Soodak [45], [68] (also in 1949). A number of references [17], [35], [36], [45], [46], [69] describe adjoints and importance in the context of neutron transport simulation.

6.2 Theoretical Results in Rendering

Pattanaik and Mudur [52], [55] and Dutré et al. [21] showed that the exitant radiance equation is the basis for light gathering methods, such as ray tracing, path tracing, and "classic" (full matrix) radiosity, and that the incident importance equation is the basis for light shooting methods such as progressive refinement radiosity and photon particle tracing from the light sources. For gathering methods, we are given $W_{i,e}$ (for example, the image plane or a current patch of interest) and must compute $\Phi = \langle L_o \mid W_{i,e} \rangle_{\omega A'}$ using the equilibrium equation for exitant radiance $L_o = L_{o,e} + \mathcal{T}_o L_o$. For shooting methods, we are given the light source emissions $L_{o,e}$ and compute $\Phi = \langle L_{o,e} \mid W_i \rangle_{\omega A'}$ using the equilibrium equation for incident directional importance $W_i = W_{i,e} + \mathcal{T}_i W_i$.

The PhD dissertations of Arvo [1] and Veach [75] derived the light transport operators and their adjoints using very strict formalisms. Veach [74], [75] showed that, if the BSDF is not symmetric (as, for example, when refraction occurs), the scattering operator $\mathcal{S}$ is not self-adjoint and the transport operators for exitant radiance and exitant directional importance differ (and the transport operators for incident radiance and incident directional importance also differ). He formulated an integration measure that takes the index of refraction into account and showed that the scattering operator is self-adjoint using an inner product with that measure.

6.3 "Classic" Ray Tracing and Distribution Ray Tracing

In ray tracing [80], the light transported by a ray is radiance [15]. The importance of a ray is simply the fraction of its radiance that ends up in the image; for this reason, importance is often called *ray weight* or *ray contribution* in ray tracing. Knowing a priori that the color resulting from tracing a ray will contribute very little to the final image makes it possible to speed up the ray tracing by making shortcuts and approximations. For example, Hall and Greenberg [27] avoided tracing reflection and refraction rays with low importance, while Arvo and Kirk [2] suggested using Russian roulette when the importance is low (to avoid introducing bias by path truncation). Cook et al. [16] suggested that the number of new rays to be traced at nonspecular reflection and refraction should be proportional to the importance, while Ward et al. [79] suggested that the number of new rays should be proportional to the square of the importance. The correct choice probably depends on the expected coherency in the samples. Ward [78] increased the sampling tolerance for unimportant shadow rays and Jensen [30] used importance to select an indirect illumination calculation method (lookups in a photon map for low importance and one level of distribution ray tracing for high importance). One can also use importance to speed up the calculation of procedural textures, bump maps, shading normals, and ray-surface intersections [9].

6.4 Finite Element Global Illumination

The finite element method used in computer graphics stems from the heat transfer literature. The finite elements are either 2D (representing positional variation of radiosity or power on diffuse surfaces) or 4D (representing positional and directional variation of radiance).

6.4.1 Diffuse Global Illumination ("Radiosity")

In finite element diffuse global illumination methods, the surfaces are divided into patches and the radiosity or power on each patch is represented with basis functions (constant, polynomial, wavelets, etc.). The influence of one basis function on another is called the "transport coefficient" or "form factor." The exitant diffuse transport operator is discretized into a matrix with form factors as the matrix elements. (The adjoint of a discretized transport operator is simply the transpose of its matrix.) In "full matrix radiosity" methods, the solution is found by solving a large system of linear equations.

A significant improvement is the use of hierarchical bases. *Hierarchical radiosity* [28] starts out by computing a solution for very coarse basis functions. An "oracle" then analyzes the solution and refines basis functions and interactions where necessary, based on estimated transport error and radiosity. Then, a new solution is found, the basis functions and interactions are refined again, and the process repeats until a sufficiently accurate solution has been found. Smits et al. [67] introduced the use of importance for a view-dependent hierarchical solution of the diffuse global illumination problem. They transported incident power importance at the same time (and using the same hierarchy) as radiosity and used estimated transport error, radiosity, and incident power importance to decide where to refine. This leads to a solution which is refined most in bright visible regions. (They made the observation that their importance—"incident power importance" in our terminology—needs to be transported differently than radiosity. For example, radiosity needs to be averaged when being "pulled up" in the hierarchy, while power importance should be added. This is simply because power importance is not measured per area.) Lischinski et al. [47], [48] improved the error bounds used for importance-driven refinement. Smits et al. [65], [66] extended the method with clustering and also used the importance-driven refinement for clusters.

Another popular finite element solution method is *progressive refinement radiosity* [14]. With this method, the solution is improved (refined) little by little, but there is no refinement of the finite elements. The patches with the highest unshot power are selected first to "shoot" their light to other patches, giving fast convergence and useful images early in the solution process. Bekaert and Willems [7] extended progressive radiosity to use importance. In their method, radiosity and incident power importance are propagated in alternating steps. In the importance propagation steps, the patch with the highest unshot importance is selected; in the radiosity propagation steps, the patch with the highest product of importance and unshot radiosity is selected. Their method allows incrementally changing view points: When the visible patches change, the importance sources are updated accordingly.

With the *bidirectional radiosity* method [20], the radiosity is computed for—and importance is emitted from—the directly visible patches one at a time. The order of the visible patches is selected depending on their projected screen area so that as many pixels are illuminated as quickly as possible. The solution for each patch is found by progressive refinement of radiosity and importance in a manner similar to the importance-driven progressive radiosity method described above, but no radiosity is computed for patches with low importance. Even though solutions are computed independently for one patch at a time, much information can be reused, including the form factors between patches and the approximate radiosity on non-visible patches.

Pueyo et al. [59] presented a radiosity algorithm with a more heuristic definition of importance. The importance is not an adjoint; instead, the important patches or objects can be manually tagged in the scene database. Radiosity is neither shot to nor from unimportant patches; the unimportant patches are only used for occlusion testing.

6.4.2 Glossy Global Illumination (Radiance)

Finite element methods have also been used to solve the more general problem of glossy global illumination. Importance is even more helpful in this case since radiance is only important if it is in a position *and direction* that contribute significantly to the image. Even within a simple glossy scene that is entirely visible, there are many unimportant transport paths.

Aupperle and Hanrahan [3], [4] extended the hierarchical radiosity method to handle glossy reflection. They let the finite elements represent radiance from one surface patch to another and replaced the "form factor" matrix with a matrix of transport coefficients representing the influence of one patch-to-patch radiance on another. Radiance and directional importance is transported using the same matrix: When radiance is transported along a link, directional importance is transported in the opposite direction along the same link. They used the product of estimated transport error, exitant radiance, and incident directional importance to decide where to refine the solution. ("Pushing" and "pulling" are the same for their choice of importance as for radiance.)

Schröder and Hanrahan [63] compared different wavelet bases for representing patch-to-patch radiance and importance. Bekaert and Willems [6] also used the same representation as Aupperle and Hanrahan, but used shooting for both light and importance to get an algorithm similar to progressive radiosity, but adapted for radiance. They used a combination of radiance and directional importance to decide which patch-to-patch basis function to shoot from next.

Christensen et al. [8], [10], [11], [12], [13] used a different representation of radiance: Instead of representing radiance between all pairs of patches, we represented radiance from each patch in the directions on the hemisphere above the patch. In [11], we used spherical harmonics to represent the directional variation; in later publications, we used wavelets. We noted that, with this representation, it is better to use exitant importance than incident importance since it has fewer discontinuities (most BSDFs act like a smoothing operator) and it is transported like radiance (simplifying the algorithm). For point representations of clusters of geometry, we introduced an importance quantity transported like radiant intensity (a representation of the light from a point with units [W/sr]) and called it *radiant importance* [10].

Several textbooks [15], [25], [64], [70] contain overviews of the use of importance-driven refinement for finite element solutions of diffuse and glossy global illumination.

6.5 Monte Carlo Global Illumination

Photon transport shares many characteristics with neutron transport, hence Monte Carlo global illumination is very similar to Monte Carlo neutron transport simulation.

6.5.1 Diffuse Global Illumination

In the *random walk radiosity* method, light particles (photons) are emitted from the light sources, transported through the scene, and stored at the premeshed diffuse surfaces they hit along the way. The sum of the power of the photons that hit a patch is an estimate of the incident power on that patch. Importance can be used to guide the photon particles to the locations where they will most improve the accuracy of the visible solution. Pattanaik and Mudur [51], [54], [56] used importance to determine the directions in which to focus the emission and reflection of photon particles. The importance of a patch was estimated by counting the fraction of the photons leaving that patch which eventually reached the region of importance. With this method, there is no explicit transport of importance.

Dutré and Willems [19], [23] also used premeshed diffuse surfaces, but only to store approximate incident importance. When a particle hits a visible point, the power is propagated directly to the corresponding pixel and stored there, so no light information is stored on the surface patches. The importance functions are used to guide emission and reflection of photon particles. The importance at a patch in a certain direction is increased if a photon particle leaving the patch in that direction eventually makes it to a visible point.

Pattanaik and Bouatouch [53] used a cheap estimate of the importance of each light source (called "illuminating capacity") to modify the probabilities of emission. Their method keeps a tally of how many photon particles emitted from each light source get reflected (through multiple bounces) into different regions of the scene. During an interactive walk-through of the scene, the emission of photon particles is biased toward emission from those light sources that contribute most to the illumination of the currently visible region.

Sbert [61] pointed out that, in order to minimize the variance of the radiosity solution, the number of photon particles to be emitted from each light source should be proportional to the square root of its importance (and not directly proportional to it). The reason is that the variance in the radiosity solution should be inversely proportional to the importance and the variance is reduced by one over the square root of the number of particles.

In *stochastic relaxation radiosity* (a.k.a. *stochastic ray radiosity*) [50], photon particles are shot from one patch at a time. At each iteration, each patch emits a number of particles proportional to its power. Points on each patch and emission directions are randomly chosen. Neumann et al. [49] traced both photon and importance particles and used importance to bias the emission probabilities (both origin patch and direction) such that more photon particles are shot to important parts of the scene. Their basic algorithm shoots power and power importance. As an improvement, they also used directional importance: The hemisphere above each patch is divided into eight solid angles and the incident importance is stored for each. Most particles are shot in important directions. Prikryl et al. [60] extended the method to use a hierarchical representation of radiosity (and importance).

6.5.2 General Global Illumination

The general global illumination method used by Dutré and Willems [19], [22] is similar to Pattanaik and Mudur's random walk radiosity method in that they both use random walks from the light sources and use importance to guide the emission from the light sources. However, Dutré and Willems used no tessellation of the scene to represent the light distribution: When a particle hits a visible surface, the contribution is stored directly in the corresponding pixel. Also, their method was not restricted to diffuse scenes—they used a Phong BRDF in their implementation. On the downside, their method did not use importance to guide reflection at surfaces, only to guide emission. (They later modified this method to also use importance sampling at surfaces. However, the new method used a tessellation to represent importance and was restricted to diffuse scenes, as described above in Section 6.5.1.)

Bidirectional path tracing combines light paths from light sources with importance paths toward the image plane. This method can be derived in a natural way using the framework of the exitant radiance equation and the incident directional importance equation [39], [40], [41], [75], [76]. In related work, several methods used information about the illumination directions to guide path tracing from the eye. Jensen [29] used photon directions from a photon map, Lafortune and Willems [42] used a 5D tree structure of ray origins and directions, and Szirmay-Kalos et al. [72], [73] used photon directions from photons stored on surface patches.

Peter and Pietrek [58] extended the photon map method [30] with an initial pass where importance particles are traced from the eye point. The resulting distribution of importance is then used in the photon tracing pass to guide the emission and reflection of photons. For example, few photons are emitted in unimportant directions (each photon with relatively high power). The problem with this method is that the power of "neighbor" photons in the photon map can vary a lot, resulting in noisy irradiance estimates. Keller and Wald [38] and Suykens and Willems [71] presented methods to overcome this problem: They use importance to determine only the probability of storing each photon. Few photons are stored in regions with low importance, but the photons that are stored there get a high power to compensate for the low probability. Keller and Wald increase the power of photons that are stored despite low probability, while Suykens and Willems distribute the power of a nonstored photon among the nearest previously stored photons. With these methods, important regions get a dense population of low-power photons while unimportant regions get a sparse population of high-power photons, thus avoiding mixing high and low-power photons. The disadvantage of these two methods is that using importance does not reduce the number of traced photons.

6.6 Participating Media

If a scene contains participating media such as fog, smoke, or silty water, the volumes should participate in the transport of light and importance.

Volume ray tracing, a.k.a. *ray marching* [57], is often used for direct illumination calculation and to render global illumination solutions. For each ray from the eye, the change in light along that ray is computed at many steps along the ray by adding the local irradiance and taking attenuation into account. Importance for eye rays in front-to-back ray marching is simply the remaining transparency along that ray, i.e., one minus accumulated opacity. Danskin and Hanrahan [18] suggested the use of importance to speed up ray marching: If the remaining importance along a ray is low, the steps can be longer and/or the calculation at each step can be simplified. Ray marching can be terminated when the remaining importance gets low [44]; this should be done with Russian roulette to avoid bias [18].

To our knowledge, importance has not been used for finite element simulation of participating media. For Monte Carlo methods, photons and importance particles can be emitted, scattered, and absorbed in the medium. Bidirectional path tracing was extended to participating media by Lafortune and Willems [43].

7 Conclusion and Future Work

Importance, defined as an adjoint quantity, originated in neutron transport simulation. In the last decade, it has been used to speed up many different rendering methods. In rendering, there are six commonly used types of importance (just as there are six commonly used representations of light): incident/exitant directional importance, diffuse importance, and power importance. Each type of importance is adjoint to at least one representation of light (some types of importance are adjoint to more than one representation of light using different inner products). We have shown precisely which types of importance are adjoint to which light representations, generalizing results known from the discrete case. We have also presented an overview of the many uses of importance in rendering.

One interesting direction for future research is the incorporation of perceptual importance measures. Currently, light transports are computed to an accuracy that is linearly dependent on importance. However, the human visual system is nonlinear—for example, small differences are more important in dark regions than in bright. It seems reasonable to use the computed importance in a nonlinear way. For example, the required accuracy could be increased in dark regions even if the importance is moderate.

Acknowledgments

The author thanks Philippe Bekaert for providing many insights into the history of importance and for finding several of the early references. Thanks to Mateu Sbert for explaining some of his observations regarding diffuse and power importance. Finally, thanks to Philippe, Mateu, Frank Suykens, and Julian Fong for reading draft versions of this paper and suggesting many improvements. This paper was written while the author was working for Square USA in Honolulu; thanks to everyone there.

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