Regularized Kelvinlets: Sculpting Brushes based on Fundamental Solutions of Elasticity ***Supplemental Material***

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A Derivatives of 3D Regularized Kelvinlets

We first provide the expressions for the derivatives of regularized Kelvinlets and locally affine regularized Kelvinlets in 3D.

Grab brush: By direct differentiation of Eq. (6) in the main text, we obtain the gradient of the 3D displacement field $\boldsymbol{u}_\varepsilon(\boldsymbol{r})$:

$$\begin{aligned}\nabla \boldsymbol{u}_\varepsilon(\boldsymbol{r}) = \frac{1}{r_\varepsilon^3}\left[b\left((\boldsymbol{f}^t\boldsymbol{r})\boldsymbol{I} + \boldsymbol{r}\boldsymbol{f}^t + \boldsymbol{f}\boldsymbol{r}^t\right) - a\,\boldsymbol{f}\boldsymbol{r}^t\right] \\ - \frac{3}{r_\varepsilon^5}\left[\frac{a}{2}\varepsilon^2\boldsymbol{f} + b\,(\boldsymbol{f}^t\boldsymbol{r})\boldsymbol{r}\right]\boldsymbol{r}^t.\end{aligned} \tag{1}$$

Observe that, at $\boldsymbol{x}_0$, we have $\nabla \boldsymbol{u}_\varepsilon(0) = 0$, thus indicating that the brush center moves rigidily.

Locally Affine brush: By differentiating Eq. (12) of the main text, we now obtain the gradient of the 3D displacement field $\widetilde{\boldsymbol{u}}_\varepsilon(\boldsymbol{r})$:

$$\begin{aligned}\nabla \widetilde{\boldsymbol{u}}_\varepsilon(\boldsymbol{r}) = &\frac{1}{r_\varepsilon^3}\left[b\,\left(\boldsymbol{F}^t + \boldsymbol{F} + \mathrm{tr}(\boldsymbol{F})\boldsymbol{I}\right) - a\boldsymbol{F}\right] \\ &- \frac{3}{r_\varepsilon^5}\left[\frac{a}{2}\varepsilon^2\boldsymbol{F} + b\,(\boldsymbol{r}^t\boldsymbol{F}\boldsymbol{r})\boldsymbol{I} + b\,\boldsymbol{r}\boldsymbol{r}^t\left(\boldsymbol{F} + \boldsymbol{F}^t\right)\right] \\ &- \frac{3}{r_\varepsilon^5}\left[b\,\left(\boldsymbol{F}^t + \mathrm{tr}(\boldsymbol{F})\boldsymbol{I} + \boldsymbol{F}\right) - a\boldsymbol{F}\right]\boldsymbol{r}\boldsymbol{r}^t \\ &+ \frac{15}{r_\varepsilon^7}\left[\frac{a}{2}\varepsilon^2\boldsymbol{F} + b(\boldsymbol{r}^t\boldsymbol{F}\boldsymbol{r})\boldsymbol{I}\right]\boldsymbol{r}\boldsymbol{r}^t.\end{aligned} \tag{2}$$

At the brush center $\boldsymbol{x}_0$, this expression simplifies to:

$$\nabla \widetilde{\boldsymbol{u}}_\varepsilon(0) = \frac{1}{\varepsilon^3}\left[b\left(\boldsymbol{F}^t + \boldsymbol{F} + \mathrm{tr}(\boldsymbol{F})\boldsymbol{I}\right) - \frac{5}{2}a\,\boldsymbol{F}\right]. \tag{3}$$

Twist brush: In this case, we have a skew-symmetric force matrix $\boldsymbol{F}$, which implies $\boldsymbol{F} + \boldsymbol{F}^t = 0$ and $\mathrm{tr}(\boldsymbol{F}) = 0$. This matrix can then be written as a cross product matrix $[\boldsymbol{q}]_\times$ of a vector $\boldsymbol{q}$, and the gradient at $\boldsymbol{x}_0$ simplifies to

$$\nabla \boldsymbol{t}_\varepsilon(0) = -\frac{5}{2}\frac{a}{\varepsilon^3}[\boldsymbol{q}]_\times. \tag{4}$$

Note that this gradient matrix is skew-symmetric, thus indicating that the deformation at the brush center is a (infinitesimal) rotation and can be controlled by the three DoFs provided by the vector $\boldsymbol{q}$.

Scale brush: This case has a force matrix $\boldsymbol{F} = s\boldsymbol{I}$ and then

$$\nabla \boldsymbol{s}_\varepsilon(0) = s\,(2b - a)\frac{5}{2}\frac{1}{\varepsilon^3}\boldsymbol{I}. \tag{5}$$

Therefore, the brush center $\boldsymbol{x}_0$ is locally deformed by a uniform scaling, with a single DoF.

Pinch brush: The case of a pinch brush has a symmetric and traceless force matrix, i.e., $\boldsymbol{F} + \boldsymbol{F}^t = 0$ and $\mathrm{tr}(\boldsymbol{F}) = 0$. This implies that the gradient $\boldsymbol{x}_0$ is of the form

$$\nabla \boldsymbol{p}_\varepsilon(0) = \frac{1}{2\varepsilon^3}(4b - 5a)\,\boldsymbol{F}. \tag{6}$$

As a consequence, the deformation at $\boldsymbol{x}_0$ is also symmetric with zero trace, in a total of five DoFs.

B Derivatives of 2D Regularized Kelvinlets

We can also compute the derivatives for regularized Kelvinlets and locally affine regularized Kelvinlets in 2D, following the differentiation of Eqs. (23) and (24) in the main text. The resulting expressions and their properties are very similar to the 3D case.

Grab brush:

$$\begin{aligned}\nabla \boldsymbol{u}_\varepsilon(\mathfrak{r}) = \frac{2}{\mathfrak{r}_\varepsilon^2}\left[b\left((\boldsymbol{f}^t\mathfrak{r})\boldsymbol{I} + \mathfrak{r}\boldsymbol{f}^t + \boldsymbol{f}\mathfrak{r}^t\right) - a\,\boldsymbol{f}\mathfrak{r}^t\right] \\ - \frac{2}{\mathfrak{r}_\varepsilon^5}\left[a\varepsilon^2\boldsymbol{f} + 2b(\boldsymbol{f}^t\mathfrak{r})\mathfrak{r}\right]\mathfrak{r}^t.\end{aligned} \tag{7}$$

At $\boldsymbol{x}_0$, we have $\boldsymbol{u}_\varepsilon(0) = 0$, thus indicating that the brush center is locally under a rigid deformation.

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Locally Affine brush:

$$\begin{aligned}\nabla\widetilde{\boldsymbol{u}}_\varepsilon(\mathfrak{r}) = &\frac{2}{\mathfrak{r}_\varepsilon^2}\left[b\,\left(\boldsymbol{F}^t+\boldsymbol{F}+\mathrm{tr}(\boldsymbol{F})\boldsymbol{I}\right)-a\boldsymbol{F}\right]\\ &-\frac{2}{\mathfrak{r}_\varepsilon^4}\left[a\varepsilon^2\boldsymbol{F}+2b\,(\mathfrak{r}^t\boldsymbol{F}\mathfrak{r})\boldsymbol{I}+2b\,\mathfrak{r}\mathfrak{r}^t\left(\boldsymbol{F}+\boldsymbol{F}^t\right)\right]\\ &-\frac{4}{\mathfrak{r}_\varepsilon^4}\left[b\left(\boldsymbol{F}^t+\mathrm{tr}(\boldsymbol{F})\boldsymbol{I}+\boldsymbol{F}\right)-a\boldsymbol{F}\right]\mathfrak{r}\mathfrak{r}^t\\ &+\frac{8}{\mathfrak{r}_\varepsilon^6}\left[a\varepsilon^2\boldsymbol{F}+2b\,(\boldsymbol{r}^t\boldsymbol{F}\boldsymbol{r})\boldsymbol{I}\right]\mathfrak{r}\mathfrak{r}^t.\end{aligned} \tag{8}$$

At a brush center $\boldsymbol{x}_0$, this expression simplifies to:

$$\nabla\widetilde{\boldsymbol{u}}_\varepsilon(0) = \frac{2}{\varepsilon^2}\left[b\left(\boldsymbol{F}^t+\boldsymbol{F}+\mathrm{tr}(\boldsymbol{F})\boldsymbol{I}\right)-2\,a\,\boldsymbol{F}\right]. \tag{9}$$

Twist brush: A 2D twist brush is described by a skew-symmetric force matrix $\boldsymbol{F} = t\boldsymbol{J}$, where t indicates a scalar and

$$\boldsymbol{J} = \begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}.$$

By replacing this matrix to Eq. (26) in the main text, we obtain

$$\boldsymbol{t}_\varepsilon(\mathfrak{r}) = -2\,a\,t\left(\frac{1}{\mathfrak{r}_\varepsilon^2}+\frac{\varepsilon^2}{\mathfrak{r}_\varepsilon^4}\right)\boldsymbol{J}, \tag{10}$$

and its gradient at $\boldsymbol{x}_0$ is of the form

$$\nabla\boldsymbol{t}_\varepsilon(0) = -t\,\frac{4a}{\varepsilon^2}\boldsymbol{J}. \tag{11}$$

This indicates that the deformation at the brush center is a rotation and is parameterized by a single DoF, representing the vorticity at $\boldsymbol{x}_0$. Note that the vorticity is a scalar in 2D, but a vector in 3D.

Scale brush: Similar to the 3D case, the scale brush has a force matrix $\boldsymbol{F} = s\,\boldsymbol{I}$ and is written as

$$\boldsymbol{s}(\mathfrak{r}) = 2(b-a)\left(\frac{1}{\mathfrak{r}_\varepsilon^2}+\frac{\varepsilon^2}{\mathfrak{r}_\varepsilon^4}\right)(s\,\mathfrak{r}). \tag{12}$$

Its gradient at $\boldsymbol{x}_0$ is

$$\nabla\boldsymbol{s}_\varepsilon(0) = 4\,s\,(2b-a)\frac{1}{\varepsilon^2}\boldsymbol{I}. \tag{13}$$

Therefore, the brush center $\boldsymbol{x}_0$ is locally deformed by a uniform scaling, with a single DoF.

Pinch brush: The case of a pinch brush has a symmetric and trace-less force matrix $\boldsymbol{F}$, and its displacement field is given by

$$\boldsymbol{p}_\varepsilon(\mathfrak{r}) = -2a\left(\frac{1}{\mathfrak{r}_\varepsilon^2}+\frac{\varepsilon^2}{\mathfrak{r}_\varepsilon^4}\right)\boldsymbol{F}\mathfrak{r}+4b\left[\frac{1}{\mathfrak{r}_\varepsilon^2}\boldsymbol{F}-\frac{1}{\mathfrak{r}_\varepsilon^4}\left(\mathfrak{r}^t\boldsymbol{F}\mathfrak{r}\right)\boldsymbol{I}\right]\mathfrak{r}. \tag{14}$$

The gradient of a pinch brush at $\boldsymbol{x}_0$ is expressed by

$$\nabla\boldsymbol{p}_\varepsilon(0) = \frac{4}{\varepsilon^2}\,(b-a)\,\boldsymbol{F}. \tag{15}$$

Due to the properties of $\boldsymbol{F}$, it is easy to check that the deformation at $\boldsymbol{x}_0$ is symmetric with zero trace, in a total of two DoFs.

C Symmetrized brushes

Without loss of generality, consider a plane centered at the origin and with normal vector $\boldsymbol{n}$. This plane defines a reflection matrix of the form $\boldsymbol{M} = \boldsymbol{I} - 2\boldsymbol{n}\boldsymbol{n}^t$. Then the image of a point $\boldsymbol{x}$ is given by $\boldsymbol{M}\boldsymbol{x}$. A symmetrized displacement field with force vector $\boldsymbol{f}$ and brush center $\boldsymbol{x}_0$ is defined by

$$\boldsymbol{u}(\boldsymbol{x}) = [\,\mathcal{K}_\varepsilon(\boldsymbol{x}-\boldsymbol{x}_0)+\mathcal{K}_\varepsilon(\boldsymbol{x}-\boldsymbol{M}\boldsymbol{x}_0)\,\boldsymbol{M}\,]\,\boldsymbol{f}. \tag{16}$$

Proposition: *Any symmetrized deformation generated by (16) satisfies the identity*

$$\boldsymbol{u}(\boldsymbol{M}\boldsymbol{x}) = \boldsymbol{M}\boldsymbol{u}(\boldsymbol{x}),\ \forall\boldsymbol{x}. \tag{17}$$

Proof:

First, we introduce some additional notation to make our derivation more concise. We denote $\boldsymbol{g} = \boldsymbol{x}-\boldsymbol{x}_0$ and $\boldsymbol{h} = \boldsymbol{x}-\boldsymbol{M}\boldsymbol{x}_0$, and indicate their norms by g and h, respectively. Also, we rewrite the Green's function $\mathcal{K}_\varepsilon$ in Eq. (6) of the main text as

$$\mathcal{K}_\varepsilon(\boldsymbol{r}) = \alpha_r\boldsymbol{I}+\beta_r\boldsymbol{r}\boldsymbol{r}^t, \tag{18}$$

where α_r and β_r are functions of the distance r and include the brush parameters (ε,μ,ν). Since $\boldsymbol{M}\boldsymbol{M} = \boldsymbol{I}$, we have $\|\boldsymbol{M}\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for any $\boldsymbol{x}$. Moreover we can show that:

$$\begin{cases}\boldsymbol{M}\boldsymbol{x}-\boldsymbol{x}_0 = \boldsymbol{M}\,(\boldsymbol{x}-\boldsymbol{M}\boldsymbol{x}_0) = \boldsymbol{M}\boldsymbol{h},\\ \boldsymbol{M}\boldsymbol{x}-\boldsymbol{M}\boldsymbol{x}_0 = \boldsymbol{M}\,(\boldsymbol{x}-\boldsymbol{x}_0) = \boldsymbol{M}\boldsymbol{g}.\end{cases} \tag{19}$$

Now replacing (16), (18), and (19) to (17), we obtain:

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{M}\boldsymbol{x}) &= [\mathcal{K}_\varepsilon(\boldsymbol{M}\boldsymbol{h})+\mathcal{K}_\varepsilon(\boldsymbol{M}\boldsymbol{g})\boldsymbol{M}]\,\boldsymbol{f}\\ &= \left[\alpha_h\boldsymbol{I}+\beta_h\boldsymbol{M}\boldsymbol{h}\boldsymbol{h}^t\boldsymbol{M}+\alpha_g\boldsymbol{M}+\beta_g\boldsymbol{M}\boldsymbol{g}\boldsymbol{g}^t\boldsymbol{M}\boldsymbol{M}\right]\boldsymbol{f}\\ &= \left[\alpha_h\boldsymbol{M}\boldsymbol{M}+\beta_h\boldsymbol{M}\boldsymbol{h}\boldsymbol{h}^t\boldsymbol{M}+\alpha_g\boldsymbol{M}+\beta_g\boldsymbol{M}\boldsymbol{g}\boldsymbol{g}^t\right]\boldsymbol{f}\\ &= \left[\boldsymbol{M}\left(\alpha_h\boldsymbol{I}+\beta_h\boldsymbol{h}\boldsymbol{h}^t\right)\boldsymbol{M}+\boldsymbol{M}\left(\alpha_g\boldsymbol{I}+\beta_g\boldsymbol{g}\boldsymbol{g}^t\right)\right]\boldsymbol{f}\\ &= \boldsymbol{M}\,[\mathcal{K}_\varepsilon(\boldsymbol{h})\boldsymbol{M}+\mathcal{K}_\varepsilon(\boldsymbol{g})]\,\boldsymbol{f}\\ &= \boldsymbol{M}\boldsymbol{u}(\boldsymbol{x}),\end{aligned}$$

thus verifying the proposition. □

A similar result is also valid for the symmetrized version of locally affine regularized Kelvinlets and for their counterparts in 2D.

D Strain-Stress Formulation

In Section 3 of the main text, we recapped the formulation of the potential energy for linear elastostatics using displacement fields $\boldsymbol{u}$. Next we review an alternative yet equivalent formulation of the elastic potential energy in terms of strain and stress tensors. Both expressions are known results in the theory of linear elasticity, see, e.g., [Slaughter 2002].

In linear elasticity, the strain tensor ϵ and the stress tensor σ are defined in terms of the displacement field $\boldsymbol{u}$ by:

$$\begin{cases} \epsilon = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right), \\ \sigma = \mu\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t\right) + \frac{2\mu\nu}{(1-2\nu)}\left(\nabla \cdot \boldsymbol{u}\right)\boldsymbol{I}, \end{cases} \tag{20}$$

where μ and ν are the elastic shear modulus and the Poisson ratio, respectively. Note that both tensors are symmetric. The elastic potential energy (Eq. (1) of the main text) is then expressed by:

$$E(\boldsymbol{u}) = \frac{1}{2}\langle \epsilon, \sigma \rangle_F - \langle \boldsymbol{b}, \boldsymbol{u} \rangle, \tag{21}$$

where $\langle \cdot, \cdot \rangle_F$ indicates the Frobenius inner product for 2-tensor fields integrated over the infinite volume, and $\boldsymbol{b}$ indicates the external body forces. Using the divergence theorem, one can show that the integrated Frobenius inner product can be converted into an integrated inner product of vector fields, i.e.:

$$\langle \epsilon, \sigma \rangle_F = -\langle \boldsymbol{u}, \nabla \cdot \sigma \rangle. \tag{22}$$

Further expanding the divergence of σ yields to:

$$\nabla \cdot \sigma = \mu \Delta \boldsymbol{u} + \frac{\mu}{(1-2\nu)} \nabla(\nabla \cdot \boldsymbol{u}). \tag{23}$$

Finally, by substituting (22) and (23) into (21) and using divergence theorem once more, we reproduce Eq. (1) of the main text.

References

W. S. Slaughter. 2002. *The Linearized Theory of Elasticity.* Birkhäuser Basel.