Abstract

We present a Lagrangian method for simulating incompressible gases in 3 dimensions. Using the vorticity equation with Lagrangian particles, we create, modify and delete vortices of varying size in a manner that handles buoyancy, boundaries, viscosity and collision with deformable objects using monopoles. Our method scales linearly for parallel processing and provides user controls at varying resolutions. We address common problems of particle-based vortex methods, and provide mathematical justification for all terms. All implementation details are provided.

1 Related Work


To model different characteristics at different resolutions, multiple approaches can be combined. Navier-Stokes and Vortex Methods can be combined and applied to a selective range of scales within the same method: Selle et al. advect vorticity carried by points [Selle et al. 2005] and Pfaff et al. advect vorticity carried by sheets [Pfaff et al. 2012]. Kim et al. advect procedurally generated detail [Kim et al. 2008]. Zhang and Bridson use a voxel grid for large distance flow interaction and particles for near flow interaction to solve the Vorticity Equation [Zhang and Bridson 2014].

While it is now widely expected that a practical algorithm must guarantee stability [Stam 1999], scalability in the number of threads is also an imminent requirement for gas simulation algorithms: the work of Gibou and Chohong [Gibou and Min 2012] and McAdams et al. [McAdams et al. 2010] report that the present bottleneck of methods that solve velocity in a voxelized grid is the Poisson equation. By avoiding the Poisson equation, our method avoids the bottleneck altogether. A boost in computing performance improves the creative iteration process.

The motivating application for this work is the animation of gases for Visual Effects. Controlling the motion of a gas presents known challenges: gases usually don’t have a well defined surface and often vary dramatically over time. To produce natural looking gases, artists have to tweak the physical parameters of computational models. Expected controls include external forces, buoyancy, rigid and deformable boundaries, and viscosity. Our model provides all of those and more: user controls at arbitrary resolution, constant cost across resolutions, spatially varying resolution, and decoupled resolutions for density, dynamics and boundary conditions. Our contributions are:

- a new basis that accounts for stretching in a stable manner.
- a new model for viscosity that handles isolated particles.
- a new model for buoyancy.
- a new model for deformable boundaries.
- a new model for vortex shedding.

Our paper is organized as follows. In Section 2, we present the equations from which we derive our model. In Section 2.1, we introduce our vorticle basis. In Section 2.2, we present our vorticity stretching technique. In Section 2.3, we present our viscosity model that handles isolated particles. In Section 2.4, we present our buoyancy model. In Section 2.5, we define the pressure field for boundaries. In Section 2.6 we present our model for vortex shedding. In Section 3, we present our acceleration data structure, followed by an overview of controls in Section 4 and algorithm summary in Section 5.

Our method is implicitly incompressible as is inherent to vortex methods, and does not require enforcing incompressibility by solving a compressible pressure term. The dynamics resolution is de-
coupled from the advected density resolution, thus the density resolution can be modified while preserving the motion. Our method is purely Lagrangian with local dynamic interaction, therefore managing the simulation domain is as simple as managing points. Our most costly step is a point cloud query.

2 Dynamic Model

To obtain our equation of motion, we apply the curl operator to both sides of the following form of the Navier-Stokes Equation of a viscous incompressible Newtonian fluid, divide both sides by $\rho$, and replace $\nabla \rho$ with $\nabla \log(\rho)$

$$\frac{d\vec{v}}{dt} = \mu \nabla^2 \vec{v} + \rho \vec{F} - \nabla p$$

(1)

We obtain the following equation that we solve for motion, where the vorticity $\vec{\omega}$ is defined as the curl of the velocity $\vec{v}$,

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega} \cdot \nabla) \vec{v} + \frac{\mu}{\rho} \nabla^2 \vec{\omega} + \nabla \log(\rho) \times (\vec{F} - \frac{d\vec{v}}{dt}) + \nabla \times \vec{F}$$

(2)

Eq. (2) means that the vorticity $\vec{\omega}$ evolves over time by advecting a Lagrangian frame of reference (particle) that carry $\vec{\omega}$, and stretching $\vec{\omega}$ according to the velocity $\vec{v}$, with dynamic viscosity $\mu$, buoyancy and boundary interaction specified by density $\rho$ and external forces $\vec{F}$

$$\vec{F}(p) = \begin{cases} \vec{g} + \vec{e} & \text{if } p \text{ outside solid objects} \\ \vec{f} & \text{otherwise} \end{cases}$$

(3)

where $\vec{g}$ is the constant for gravity, $\vec{f}$ is the acceleration at the objects’ boundaries suitable for deformable objects, and $\vec{e}$ is user defined external forces. The density is assumed strictly greater than 0. The velocity $\vec{v}$ is needed for advection, and is obtained from $\vec{\omega}$ by inverting the curl operator with the Biot-Savart law and an irrotational and solenoidal field $\vec{h}$

$$\vec{u}(p) = \frac{1}{4\pi} \int_{x \in \mathbb{R}^3} \frac{\vec{\omega}(x) \times (p - x)}{||p - x||^3} \, dx$$

$$\vec{\omega} = \vec{u} + \vec{h}$$

(4)

Eq. (4) means that the flow $\vec{v}$ is the sum of the velocity induced by a continuum of rotations of center $x$, axis $\vec{\omega}$ and angle $||\vec{\omega}||/||p - x||^3$, with a pressure field $\vec{h}$ that models the boundary condition. The relation between $\vec{h}$ and $\vec{f}$ is given by Eq. (20) and the condition of Section 2.5. Our nomenclature is provided in Appendix A.

2.1 Vorticle and Stretchable Vorticle

This section introduces our basis. We keep the integral inside our discretization of Eq. (4), and introduce with Eq. (5) a vorticle partitioning $(\vec{V}_i, \vec{\omega}_i)$ where $\vec{\omega}_i$ denotes the vorticity field induced by vorticle $i$

$$\vec{u}(p) = \frac{1}{4\pi} \sum_i \int_{x \in V_i} \frac{\vec{\omega}_i(x) \times (p - x)}{||p - x||^3} \, dx$$

(5)

The singularity of Eq. (5) at $p = x$ could be removed by integrating analytically the vorticity over the partition or with a regularizing constant. To avoid this laborious integral and avoid an arbitrary post-simulation blurring filter size, we define instead the vorticity field as the sum of the curl of the vorticle’s velocity field, as shown in Figure 2, in contrast with other methods that define vorticity as interpolated from point values. This guarantees a divergence free vorticity and avoids instability from vorticity compression. A vorticle is a vorticity particle defined by rotation strength $\vec{w}_i$, center $x_i$ and falloff $\phi_i(p)$. The velocity and vorticity fields induced by a vorticle are shown in Figure 2 and given by

$$\vec{\nu}_i(p) = \phi_i \vec{w}_i \times (p - x_i)$$

(6a)

$$\vec{\omega}_i(p) = 2 \phi_i \vec{w}_i + \nabla \phi_i \times (\vec{w}_i \times (p - x_i))$$

(6b)

The falloff of a stretchable vorticle $\phi_i = s_i r_i^{-5/2}(1 + \mu_s/(p - x_i)/r_i)^{-3/2}$ and its gradient $\nabla \phi_i$ are centered at $x_i$, where $r_i$ is the vorticle’s size, $s_i$ is the stretching factor, and $\mu_s$ is the stretching function $\mu_s(q) = \frac{1}{w^2} (s_i^{-4/5}(\vec{w}_i \cdot q) \vec{w}_i + s_i^{7/10} \vec{w}_i \times q \times \vec{w}_i)$. The following properties are satisfied by $\phi_i$. First, $\phi_i$ revolves around $\vec{w}_i$, and therefore $\vec{\nu}_i$ is divergence-free since its magnitude is constant along the streamlines of the rotation. Second, when the stretching factor is increased from $s_i$ to $s_i'$, the vorticity $\vec{\omega}_i$ is stretched by a factor $s_i'/s_i$ along $\vec{w}_i$ and squashed by $\sqrt{s_i/s_i'}$ along any direction perpendicular to $\vec{w}_i$, in accordance with the deformation induced by an incompressible flow, and satisfying Kelvin’s circulation theorem. Third, stretching is conservative since the vorticle’s mean energy $E_i$ is independent of $s_i$.

$$E_i = \frac{1}{2} \int \vec{\nu}_i^2 \, dx = \sqrt{2\pi^2} ||\vec{w}_i||^2$$

(7)

A stretchable vorticle is thus defined by 4 variables $\{x_i, \vec{w}_i, r_i, s_i\}$, and we show in the next section how to handle $s_i$ and reduce it to a vorticle of 3 variables

$$\{x_i, \vec{w}_i, r_i\}$$

(8)

The velocity field is defined by summing the velocity field of multiple vorticles, as shown in Figure 3.

![Figure 2: Left: slice of the velocity field induced by a vorticle. Right: slice of the vorticity field induced by a vorticle.](image)

![Figure 3: A velocity field is induced by multiple vorticles, illustrated with three vorticles orthogonal to a plane.](image)
2.2 Stretching

The term $(\vec{\omega} \cdot \nabla) \vec{v}$ in Eq. (2) models the stretching of vorticity. We measure stretching at $x_i$ by applying the velocity gradient to $\vec{\omega}, (x_i) = 2r_i^{-5/2}\vec{w}_i$, the self-induced vorticity at the center of an unstretched vorticle. We obtain

$$\frac{\partial \vec{\omega}_i}{\partial t} (x_i) = \sum_j \vec{w}_j \times \left( \nabla \phi_j (x_i) \cdot \vec{\omega}_i (x_i - x_j) + \phi_j (x_i) \vec{\omega}_i (x_i) \right)$$

Stretching produces a rotation of $\vec{w}_i$ and scale of $s_i$. Let us call $\vec{\omega}_i' = \vec{\omega}_i + D \frac{\partial \vec{\omega}_i}{\partial t}$. The new rotation strength and stretch factor are:

$$\vec{w}_i' = \frac{\vec{w}_i \| \vec{\omega}_i' \|}{\| \vec{\omega}_i \|} (10)$$

$$s_i' = \frac{\| \vec{w}_i' \|}{\| \vec{\omega}_i \|}$$

The accumulation of stretching introduces increasingly high frequency velocities by transferring large scale eddies to smaller scale eddies [Frisch and Kolmogorov 1995]. Although the diffusion of Section 2.3 filters eddies over long enough periods of time, instability is not an option and high frequency eddies must be filtered explicitly in a predictable manner. We filter by unstretching the stretchable vorticle in a manner that preserves both $E_i$ and the enstrophy $\Omega_i$ of the stretched vorticle

$$\Omega_i = \frac{1}{2} \int \vec{\omega}_i^2 dx = \frac{3\pi^2 (1 + 4s_i^2)}{162r_i^{5/2}} \| \vec{w}_i \|^2$$

Preserving $E_i$ is trivial since $E_i$ is independent of $s_i$ and $r_i$ in Eq. (7). To preserve $\Omega_i$, we adjust $r_i$ to a new size $r_i'$

$$r_i' = r_i s_i'^{4/5} \sqrt{\frac{5}{1 + 4s_i'^2}}$$

With Eq. (13) the stretching factor can be restored to 1. This is illustrated in Figure 4. The reader can verify that swapping $\{r_i, s_i'\}$ for $\{r_i, 1\}$ preserves $\Omega_i$. This step is a resampling step, where we use the same vorticle locations. Note that resampling introduces an error, especially when squashing i.e. the stretching factor is below 1. Therefore if taking substeps, we recommend unstretching on full frames to avoid overfiltering. We also set limit resolutions with a lower threshold $r_{min}$ and upper threshold $r_{max}$ on the vorticle size $r_i$. This limit resolution loses enstrophy, but does not lose energy because of Eq. (7). Thus Eq. (10), Eq. (11) and Eq. (13) provide the way to apply stretching to a vorticle. The falloff and falloff gradient for an unstretched vorticle are

$$\phi_i = \sqrt{r_i (r_i^2 + \| p - x_i \|^2)^{-3/2}}$$

$$\nabla \phi_i = -\frac{3}{2} \sqrt{r_i (r_i^2 + \| p - x_i \|^2)^{-5/2}} (p - x_i)$$

When a vorticle becomes too small and approaches the fluid’s Kolmogorov length, viscous forces dominate, and the vorticle strength is dissipated with $\vec{w}_i' = k \vec{w}_i$.

2.3 Viscosity

The term $\frac{\nu}{\rho} \nabla^2 \vec{\omega}$ in Eq. (2) is the diffusion of vorticity, and models viscosity $\nu$. We approximate $\frac{\nu}{\rho}$ with a constant kinematic viscosity $\nu$, and derive our viscous model from the modified PSE (particle-strength exchange) method described by Cottet and Koumoutsakos [Cottet and Koumoutsakos 2000] which normalizes the discrete integral to avoid a blow up, and we add a term for handling isolated particles to model effectively the leakage of vorticity into the region of space with no vortices. The PSE method is obtained by a Taylor expansion of $\vec{\omega}$, reduced after multiplication with a normalized regularization function $\eta_i$. The result of this term is similar to the artificial damping of Park and Kim [Park and Kim 2005], but within the scope of the diffusion

$$\frac{\partial \vec{\omega}_i}{\partial t} = \frac{\nu}{\rho} \left( \frac{1}{2} \sum_j \eta_j (x_i - x_j) \nabla \rho \bullet \vec{w}_j - \alpha_i \vec{w}_i \right)$$

where $\alpha_i$ is a measure of the isolation of particle $i$, $\eta_i$ is the Gaussian PSE kernel, and $V_i$ is the volume associated with vorticle $i$

$$\eta_i (x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{\| x - x_i \|^2}{2\sigma^2} \right)$$

$$V_i = \int \phi_i dV$$

We use $\nu = \sqrt{\nu D_i}$, thus making viscosity cheaper for low $\nu$ and small time steps. This result can also be interpreted as a convolution with the heat kernel.

2.4 Buoyancy

When $p$ is outside of solid objects, the term $\nabla \log (\rho) \times (\vec{F} - \frac{\vec{\omega} \cdot \vec{F}}{\rho} \vec{F}) + \nabla \times \vec{F}$ in Eq. (2) reduces to $\nabla \log (\rho) \times (\vec{g} + \vec{\epsilon} - \frac{\alpha_i}{\rho} \vec{v})$, and models the vorticity induced by buoyancy. New vortices are produced from the density field $\rho$ releasing potential energy, as shown in Figure 5. We define $\rho$ with a set of particles carrying density and an ambient density $\rho_\lambda > 0$, so the total density field $\rho$ is strictly greater than 0, as required by Eq. (2)

$$\rho = \rho_\lambda + \sum_j \rho_j$$

We use subscript $j$ to denote density particles, as opposed to subscript $i$ for vortices. The falloff $\rho_j$ of a density particle $j$ is centered at $x_j$, and defined per Appendix B in local coordinates $q_j = p - x_j$

$$\exp \left( (1 + \kappa_1 \frac{q_j}{2\kappa_2^2})^{-\kappa_2} - 1 \right)$$

where $m_j$ is a multiplier of the particle density field, and $\kappa_1$ is given in Eq. (37). The newly induced vortices are located along a ring of diameter $r_j$ perpendicular to $\vec{g} + \vec{\epsilon} - \frac{\alpha_i}{\rho} \vec{v}$. Instead of advecting an additional filament representation, we discretize the ring with $n$ new vortices, equidistant for simplicity, and where $n = 2$ in practice. Let us define orthogonal unit vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a} \times \vec{b}$ has the direction of $\vec{g} + \vec{\epsilon} - \frac{\alpha_i}{\rho} \vec{v}$. Given $n$ randomly selected
2.5 Boundary Pressure

In Eq. (4), the field $\mathbf{h}$ cancels the velocity field induced by the vortices without adding vorticity. To define $\mathbf{h}$, we split $\mathbb{R}^3$ with a surface $\delta\Omega$ enclosing volume $\Omega$ with normal $\mathbf{n}$ pointing outside, and volume $\Omega^c$ the complementary of $\Omega$. We show in Appendix C that the Neumann boundary condition defines $\mathbf{h}$ restricted to $\Omega^c$ explicitly as

$$\mathbf{h}_{\Omega^c}(\mathbf{p}) = \iiint_{\delta\Omega} \mathbf{n} \cdot \left( \mathbf{v}_{\Omega^c}(\mathbf{x}) - \mathbf{u}(\mathbf{x}) \right) \nabla G(\mathbf{p}, \mathbf{x}) \, d\mathbf{x} \tag{20}$$

Here $G(\mathbf{p}, \mathbf{x}) = -\frac{1}{4\pi|\mathbf{p} - \mathbf{x}|^3}$ is the Green’s function of the Laplacian, $\mathbf{v}_{\Omega^c}$ is the velocity of the boundary. To the best of our knowledge, we are not aware of this result in previous work, although this is related to the method of Zhang and Bridson [Zhang and Bridson 2014] who solve $\mathbf{h}$ numerically using the single layer potential from Potential theory. Using $n$ panels of size $a_i$ and centroid $x_i$, we could discretize $\mathbf{h}$ using directly the falloffs

$$\mathbf{h}_{\Omega^c}(\mathbf{p}) = \sum_i a_i \mathbf{n}_i \cdot \left( \mathbf{v}_{\Omega^c}(x_i) - \mathbf{u}(x_i) \right) \nabla G(\mathbf{p}, x_i) \tag{21}$$

To avoid the singularities of $G$ when $\mathbf{p}$ approaches the boundary samples $x_i$, we use, instead of Eq. (21), the pathlines of the monopole based on the deformer of Decaudin [Decaudin 1996]. For a point outside the boundary

$$\mathbf{h}_{\Omega^c}(\mathbf{p}) = \frac{1}{D_t} \sum_i \zeta(\mathbf{p} - x_i, D_t a_i \mathbf{n}_i \cdot \left( \mathbf{v}_{\Omega^c}(x_i) - \mathbf{u}(x_i) \right)) \, \, \, - \mathbf{p} - x_i$$

where $\zeta$ defined below satisfies $\zeta(\mathbf{p}, \mathbf{k}_0) = \zeta(\mathbf{p}, \mathbf{k}_0 + \mathbf{k}_1)$

$$\zeta(\mathbf{p}, \mathbf{k}) = \begin{cases} (1 + \frac{3}{4\pi|\mathbf{p}|^3})^{1/3} & \text{if } r(\mathbf{k}) < |\mathbf{p}| \\ 0 & \text{otherwise} \end{cases}$$

$$r(\mathbf{k}) = (\max(-k, 0) \frac{3}{4\pi})^{1/3} \tag{23}$$

This provides a geometric insight: a boundary opposing the flow is akin to an insertion and removal of volume at the boundary proportional to the boundary’s opposition to the flow. The discretization of $\mathbf{h}$ is stable, but more accurate away from the boundary than near the boundary. To remedy the problem of accuracy, we define the solution $\mathbf{h}_{\Omega^c}$ on the boundary. Since $\mathbf{n} \cdot \mathbf{h}_{\Omega^c} = \mathbf{n} \cdot (\mathbf{v}_{\Omega^c} - \mathbf{u})$ and $\mathbf{h}_{\Omega^c}$ is aligned with the normal.

$$\mathbf{h}_{\Omega^c}(\mathbf{p}) = \arg\min_{\mathbf{w}\in\Omega^c} (\mathbf{w} - \mathbf{p}) \tag{24}$$

And finally, if a point $\mathbf{p}$ enters the boundary, it is pushed out to the nearest position on the surface

$$\mathbf{h}_{\Omega^c}(\mathbf{p}) = \arg\min_{\mathbf{w}\in\Omega^c} (\mathbf{w} - \mathbf{p}) \tag{25}$$

We assemble Eq. (22), Eq. (24) and Eq. (25) to construct the full definition of $\mathbf{h}$, by blending $\mathbf{h}_{\Omega^c}$ and $\mathbf{h}_{\Omega^c}$ with a smoothstep function based on the distance $\hat{r}$ between the samples on the boundary

$$\mathbf{h}_{\Omega^c}(\mathbf{p}) = \begin{cases} \mathbf{h}_{\Omega^c}(\mathbf{p}) & \text{if } \mathbf{p} \Omega^c \\ \min(\mathbf{h}_{\Omega^c}(\mathbf{p}), \mathbf{h}_{\Omega^c}(\mathbf{p})) & \text{otherwise} \end{cases} \tag{26}$$

Where mix($\mathbf{a}, \mathbf{b}$) = $\mathbf{a} + \text{smoothstep}(\hat{r})(\mathbf{b} - \mathbf{a})$, given the distance $d$ from $\mathbf{p}$ to $\delta\Omega$.

2.6 Vortex Shedding

When $\mathbf{p}$ is at the boundary of a solid object, the term $\nabla \log(\rho) \times (\mathbf{F} - \frac{\partial \mathbf{u}}{\partial \mathbf{r}}) + \nabla \times \mathbf{F}$ in Eq. (2) measures the change of vorticity at the moving object’s boundary. This vorticity spreads into the flow by a diffusion proportional to viscosity coefficient $\nu$ introduced in Section 2.3. We show in Appendix D that the surface vorticity that satisfies our boundary conditions is $(\mathbf{F} - \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{r}}) \times \mathbf{n}$. The vorticity shedding is the solution to a differential equation with boundary condition

$$\begin{cases} \frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega \\ \omega = \frac{\mathbf{n}}{\nu \delta\Omega} \mathbf{D}_t (\mathbf{F} - \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{r}}) \times \mathbf{n} \end{cases} \tag{27}$$

We solve Eq. (27) by shedding vortices: we divide the surface into $n$ panels of area $a_i$ and centroid $x_i$, and emit per panel a vorticle that approximates the heat kernel

$$\begin{align*}
\mathbf{w}_i &= a_i \frac{\nu^{3/2}}{\sqrt{2\pi \nu D_t}} (\mathbf{F} - \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{r}}) \times \mathbf{n} \\
r_i &= \sqrt{2\pi \nu D_t} 
\end{align*} \tag{28}$$

We use $||\mathbf{F} - \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{r}}||$ as a probability distribution function to create samples at the locations that most affect the flow. Note that $\mathbf{F}$ is the surface acceleration, as opposed to the velocity. To compute the acceleration, we store on the surface Eq. (4) at the previous time, and use $\frac{\partial \mathbf{u}}{\partial \mathbf{r}} \simeq \frac{\mathbf{u}(t) - \mathbf{u}(t-D_t)}{D_t}$. We show in Figure 7 that this produces the expected behavior.
Then the remapped falloff and gradient are computed, with defaults typically set to the vorticle cutoff. Vorticles that are similar and close enough can be fused in one vorticle. Vorticles are similar enough when their radii are equal to the sum when the vorticles have the same radius. Life-span 

For efficiency, we introduce a vorticle cutoff proportional to the vorticle size \( r_v \). This reduces the algorithmic complexity to \( O(N) \) when neighbor cell lists are used. This accelerates the nearest vorticle search while preserving an incompressible flow since the cutoff is constant along the streamlines of rotation. Since vorticles are band limited, we can further split the vorticles into groups of similar radius, and evaluate their displacement in sparse grids with cell size proportional to the cutoff.

**Vorticle cutoff** We introduce the cutoff on both the falloff and its gradient in a manner that preserves smoothness. Let us define a vorticle’s cutoff distance \( mr_v \). Using the following values

\[
\begin{align*}
    k_0 &= 4(1 + 2m^2)r_i \\
k_1 &= 2 + m^2 \\
k_2 &= k_1^2 / k_3 \\
a_0 &= \frac{r_i}{g_2 G_2} \\
a_1 &= \sqrt{\frac{2}{20 - 6m^2}} \frac{\|p - x_i\|}{k_2} \\
a_3 &= \frac{\xi}{k_2} / \sqrt{r_i^2}
\end{align*}
\]

(29)

Then the remapped falloff and gradient are

\[
\begin{align*}
    \tilde{\phi}_i &= a_0 (\phi_i - a_1) \\
    \vec{\nabla} \tilde{\phi}_i &= \nabla \phi_i + a_3 (p - x_i)
\end{align*}
\]

(30)

The vorticle cutoff provides an adjustable tradeoff between the falloff’s physically based properties and spatially localized computations, with defaults typically set to \( 6r_i \) and \( r_i/2 \).

**Fusing vorticles** Vorticles that are similar and close enough can be fused in one vorticle. Vorticle are similar enough when \( \|p_i - p_j\| \) and \( |r_i - r_j| \) are smaller than a distance threshold \( \epsilon \). Also, the velocity induced by a group of far away vorticles can be obtained by fusing the vorticles. The fused vorticle is given by the following formula

\[
\begin{align*}
    x &= \frac{\sum_i x_i}{n} \\
r &= \left( \frac{\sum_i r_i^{-5/3}}{n} \right)^{-1/3} \\
\tilde{\vec{w}} &= \frac{\sum_i \sqrt{\tilde{\phi}_i} \vec{w}_i}{\sqrt{n}}
\end{align*}
\]

(31)

where \( n \) is the number of vorticles in the cell and \( i \) is the vorticle index. The above is asymptotic to the sum of vorticles, and exactly equal to the sum when the vortices have the same radius.

**Lifespan** Vorticles can be assigned an artificial decay rate, triggered by an event controlled by a varying expression.

**Band-limiting** As stretching deforms vorticles outside of the range \( (r_{\text{min}}, r_{\text{max}}) \), we reduce the vorticle’s strength \( \tilde{\vec{w}}_j \) instead of scaling the vorticle’s radius \( r_v \). For shrinking small vorticles, this models the fluid’s viscous behavior at small scales, and for expanding large vorticles this reduces their contribution to velocity artificially.

**Radius paging** Splitting the vorticles in groups of similar radius leads to a more efficient use of acceleration structure for distance queries. We use a logarithmic scale to split the vorticles in groups.

**Cached velocity** When the point density is particularly high within a region of space, velocity can be evaluated at the vertices of a lattice and interpolated in-between.

![Figure 7](image7.png) **Figure 7:** Left: a constant flow moves at \( 10 \) m/s with \( \nu = 0.1 \) around a static pole of radius 1 (red): the surface sheds vorticles (black arrows). Right: a slice of the vorticity induced by the vorticles reveals the emergent behavior of a von Karman vortex street.

**Figure 8:** Left: 2400 vorticles, showing the vertical current and the collider’s trail. Right: advected densities. Dynamics can be limited to the visible areas simply by deleting points.

4 Controls

The number of samples of the colliders, shedding, sources, buoyancy and density are controlled independently. Users can control the flow by modifying existing vorticles with external forces \( \vec{f} \) or via the harmonic field \( \vec{h} \), or by creating new vorticles:

- A turbulent field is created by scattering vorticles with random parameters.
- A gust of wind is created by placing vorticles aligned with the tangent of a ring perpendicular to the direction of the desired wind.
- An invisible collider can be moved to warp space.
- The amount of shedding and buoyancy can be artificially dialed up or down.

A number of additional dials can be tuned per vorticle:

**Overshoot** For advection of visible particles, the velocity can be scaled lower or higher to create a drag or extra swirliness. Although physically implausible, this helps creating styles.

**Spin** The axis of vorticles can be aligned with vector fields to modify the fluid motion globally.

**Artificial damping** The strength of vorticles can be reduced artificially. This damping coefficient can be stored per vorticle.

5 Algorithm Summary

In this section we summarize our algorithm. Several integration schemes may be used to advect particles along the induced velocity...
field. A simple method, very stable for large time steps, relies on the well understood circular pathlines of individual vorticles, and we replace Eq. (6a) with
\[
\vec{K}_w = D_t \phi_i (\vec{p} - \vec{x}_i) \vec{w}_i, \\
\vec{w}_i = \vec{K}_w \times (\vec{p} - \vec{x}_i), \\
\vec{v}_i (\vec{p}) = \frac{1}{D_t} \left( 1 - \cos\left( \frac{||K_w||}{k_w} \right) \right) \vec{K}_w \times \vec{w}_i + \sin\left( \frac{||K_w||}{||K_w||} \right) \vec{w}_i \tag{32}
\]

In the following we distinguish between density particles for buoyancy, vortices for dynamics, and visual particles for rendering.

- Scatter point samples on boundaries using the probability distribution of Section 2.6.
- Compute and store the velocity \( \vec{u} \) induced by vortices on boundary point samples using Eq. (4).
- Create new vortices from vortex shedding, defined by Eq. (28) of Section 2.6.
- Create new vortices from buoyancy, defined by Eq. (19) of Section 2.4.
- Emit new vortices as defined by the user. Scattering vortices produces an initial condition. Emitting vortices on a circle with tangential strength produces a source of wind.
- Build a hierarchy by fusing vortices, to accelerate evaluating the velocity induced by groups of distant vortices.
- Compute and store the velocity \( \vec{H} \) induced by colliders on the visual particles using Eq. (26) of Section 2.5. Compute and store the velocity \( \vec{H} \) induced by vortices on the visual particles using Eq. (4).
- Compute and store the velocity \( \vec{H} \) induced by colliders on the density particles using Eq. (26) of Section 2.5. Compute and store the velocity \( \vec{H} \) induced by vortices on the density particles using Eq. (4).
- Apply displacement \( D_t (\vec{u} + \vec{H}) \) to visual particles, density particles and vortices.
- Fuse vortices that can be fused using Eq. (31).
- Attenuate vortices.
- Delete vortices far from visual particles, outside of frustum, or with a low strength.

6 Conclusion

We presented a method for simulating gases using vortices. Our method has no divergence by construction and avoids solving pressure completely. The simulation is gridless and can span domains of arbitrary size and resolution. The sampling resolution of density, dynamics, boundary condition and sources are decoupled from each other, which opens many options for adaptive sampling strategies. Although our method is similar to other point-based vortex methods, a subtle difference is that instead of filtering a point vorticity, we advect vortices that carry a volume of vorticity. This leads to stable stretching. Additional stability is achieved by integrating streamlines instead of velocity. Also, we propose a model for boundary pressure that does not require solving a linear system, and we propose a model for buoyancy which plays a key role in

\[
\begin{align*}
t & \quad \text{time} \quad D_t & \quad \text{time step} \\
p & \quad \text{position} \quad \vec{x}_i & \quad \text{vorticle center} \\
\vec{v} & \quad \text{velocity} \quad \vec{v}_i & \quad \text{vorticle velocity} \\
\vec{\omega} & \quad \text{vorticity} \quad \vec{\omega}_i & \quad \text{vorticle vorticity} \\
\vec{w}_i & \quad \text{vortex strength} \quad r_{i} & \quad \text{vortex size} \\
s_i & \quad \text{vortex stretching factor} \quad \rho & \quad \text{density} \\
\rho & \quad \text{density} \quad \rho_A & \quad \text{ambient density} \\
\rho_j & \quad \text{particle density} \quad m_j & \quad \text{particle density multiplier} \\
\nu & \quad \text{kinematic viscosity} \quad \mu & \quad \text{dynamic viscosity} \\
E_i & \quad \text{vortex mean energy} \quad \Omega_i & \quad \text{vortex enstrophy} \\
\vec{F} & \quad \text{external forces} \quad \delta & \quad \text{solid objects acceleration} \\
\vec{g} & \quad \text{gravity} \quad \Omega & \quad \text{space inside boundary} \\
\Omega^c & \quad \text{space outside boundary}
\end{align*}
\]

B Buoyancy

The vorticity induced by buoyancy could be computed directly by sampling everywhere in \( \mathbb{R}^3 \), the buoyancy term with vortices. But this laborious integral can be avoided, since buoyancy vorticity is concentrated near places of varying density and where the density gradient and buoyant acceleration are perpendicular. First we associate a set of vortices with a single particle \( j \) which we will generalize to multiple density particles. Let us define a local coordinate system centered at \( \vec{x}_j \) such that \( \vec{Z} \) is aligned with \( \vec{g} + \vec{e} = \vec{e} \). For a density particle or radius \( r_j \), we expect the vortices induced by buoyancy to be concentrated around a ring \( \{ \vec{x}_j, \alpha \in 2\pi \} \) of diameter \( r_j \), perpendicular to vector unit \( \vec{Z} \), with an axis \( \vec{n}_i \). In local coordinates

\[
\begin{align*}
\vec{x}_j & = \frac{r_j}{2} \left( \cos(\alpha), \sin(\alpha), 0 \right) \\
\vec{n}_j & = \left( -\sin(\alpha), \cos(\alpha), 0 \right)
\end{align*}
\]

For a canonical density particle, we use the following density field

\[
\rho_j = \exp\left( \left( 1 + \kappa_1 \frac{\vec{X} \cdot \vec{X}}{2r_j^2} \right)^{-\kappa_2} \right) \tag{35}
\]

We fit parameters \( \kappa_0, \kappa_1 \) and \( \kappa_2 \) using the following equation

\[
\begin{align*}
\nabla \log(\hat{\rho}_j) \times \vec{Z} & = r_j \sqrt{r_j \kappa_0} \nabla \left( \int_{\alpha} \varphi(\vec{X} - \vec{x}_j) \vec{w}_\alpha \times (\vec{X} - \vec{x}_j) d\alpha \right)
\end{align*}
\]

We obtain the following parameters. The left and right side of Eq. (36) are compared in Figure 9

\[
\begin{align*}
\kappa_0 & = 0.493483 \\
\kappa_1 & = 0.572636 \\
\kappa_2 & = 3.423340
\end{align*}
\]

To generalize to multiple particles, we remap \( \hat{\rho}_j \) to values in \( (0, m_j) \) so that densities can be added, and obtain the field of a single particle density \( \rho_j = m_j \frac{\vec{X} \cdot \vec{Z}}{2r_j^2} \). Since \( \frac{\vec{X} \cdot \vec{Z}}{r_j^2} \times \vec{Z} = \sum m_j \frac{\vec{X}_j \cdot \vec{Z}}{r_j^2} \nabla \varphi(\vec{X} - \vec{x}_j) \), we obtain a set of weights that modulate the rotations of the vortices induced by the density particle \( j \), that accounts for acceleration and multiple particles

\[
\vec{w}_\alpha = r_j \sqrt{r_j \kappa_0} \frac{\pi}{n} \left( \vec{e} - \frac{d\vec{v}}{dt} - \frac{m_j}{\rho_j} \frac{\vec{v}_j}{\rho_j} \right) \vec{n}_\alpha \tag{38}
\]
C Boundary Pressure

Let us call $G$ the Green’s function of the Laplacian $G(p, x) = \frac{1}{4\pi|p - x|^2}$. Let us denote $\Omega$ a surface enclosing volume $\Omega$, with normal $\vec{n}$ pointing outside, and volume $\Omega^c$ the complementary of $\Omega$.

Lemma 1 For any vector field $\vec{v}$ we can define the following irrotational and incompressible vector field $\vec{h}_{\Omega^c}$, such that $\vec{v} + \vec{h}_{\Omega^c}$ does not cross the surface $\partial \Omega$:

$$\vec{h}_{\Omega^c}(p) = -\iint_{\partial \Omega} \vec{n} \cdot \vec{v}(x) \nabla G(p, x) \, dx$$  \hspace{1cm} (39)

Proof 1 Using the delta function

$$\delta^3(p - x) = \nabla^2 G(p, x)$$  \hspace{1cm} (40)

and using the vector derivative identity

$$\nabla \cdot \vec{F} = \nabla \cdot (\vec{F} \cdot \vec{n}) - \nabla \times (\vec{F} \times \vec{n})$$  \hspace{1cm} (41)

for $p \in \Omega^c$ we write the Helmholtz decomposition of $\vec{h}_{\Omega^c}$, as the sum of an irrotational field and solenoidal field

$$\vec{h}_{\Omega^c}(p) = \iint_{\Omega^c} \vec{h}_{\Omega^c}(x) \delta^3(p - x) \, dx$$

$$\begin{align*}
&= \nabla \iint_{\Omega^c} \nabla \cdot (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx \\
&= \nabla \iint_{\Omega^c} \nabla \cdot (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx \\
&= \nabla \iint_{\Omega^c} \nabla \cdot (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx
\end{align*}$$  \hspace{1cm} (42)

With derivative notation $\nabla = \frac{d}{d\vec{x}}$, we note that $\nabla G(p, x) = -\nabla G(p, x)$

$$\vec{h}_{\Omega^c}(p) = \nabla \iint_{\Omega^c} \nabla \cdot (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx$$

Then since $\vec{h}_{\Omega^c}$ is solenoidal and irrotational we can use identities $\nabla \cdot \vec{v} = \nabla \cdot (\vec{G} \vec{h}_{\Omega^c})$ and $\nabla \times \vec{v} = \nabla \times (\vec{G} \vec{h}_{\Omega^c})$ to write

$$\begin{align*}
\vec{h}_{\Omega^c}(p) &= -\nabla \iint_{\Omega^c} \nabla \cdot (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx \\
&+ \nabla \iint_{\Omega^c} \nabla \times (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx
\end{align*}$$  \hspace{1cm} (43)

Then using the divergence theorem of the first term, assuming $\vec{n}$ points outside and the field vanishes faster than $1/r$ as $r \to \infty$

$$\begin{align*}
\vec{h}_{\Omega^c}(p) &= \nabla \iint_{\Omega^c} \nabla \cdot (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx \\
&+ \nabla \iint_{\Omega^c} \nabla \times (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx
\end{align*}$$  \hspace{1cm} (44)

Field $\vec{h}_{\Omega^c}$ is irrotational, and if $\Omega^c$ is simply connected we can then write $\vec{h}_{\Omega^c}$ as the gradient of a scalar field, $\vec{h}_{\Omega^c} = \nabla \phi$. Using the identity $\nabla \times (G \nabla \phi) = -\nabla \times (\nabla \times (\nabla G))$ on the second term

$$\begin{align*}
\vec{h}_{\Omega^c}(p) &= \nabla \iint_{\Omega^c} \nabla \cdot (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx \\
&- \nabla \iint_{\Omega^c} \nabla \times (\nabla G(p, x)) \, dx
\end{align*}$$  \hspace{1cm} (45)

Then using the curl theorem of the second term, assuming the field vanishes faster than $1/r$ as $r \to \infty$

$$\begin{align*}
\vec{h}_{\Omega^c}(p) &= \nabla \iint_{\Omega^c} \nabla \cdot (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx \\
&- \nabla \iint_{\Omega^c} \nabla \times (\nabla G(p, x)) \, dx
\end{align*}$$  \hspace{1cm} (46)

We apply the gradient to the first term and the curl to the second term

$$\begin{align*}
\vec{h}_{\Omega^c}(p) &= \nabla \iint_{\Omega^c} \nabla \cdot (G(p, x) \vec{h}_{\Omega^c}(x)) \, dx \\
&- \nabla \iint_{\Omega^c} \nabla \times (\nabla G(p, x)) \, dx
\end{align*}$$  \hspace{1cm} (47)

Since the integrands are irrotational and solenoidal, so is their superposition by the integral. If function $\chi$ takes a constant value on the surface of the obstacle, say 0, then $\nabla \chi$ is aligned with $\vec{n}$ and we can impose the Neumann boundary condition $\vec{v} + \vec{h}_{\Omega^c} = 0$ on $\partial \Omega$. Note that if surface $\partial \Omega$ is topologically not simply connected, then $\chi$ is only locally defined, but since we set $\chi = 0$ on the boundary, we can reconnect the branches of $\chi$.

D Boundary Vorticity

The vorticity near the boundary $\Omega$ of a moving object is given by $\nabla \log(\rho) \times (\vec{F} - \vec{a}) + \nabla \times \vec{F}$, where $\vec{a}$ is the acceleration near the boundary. To compute this term, we consider a coordinate system where the half space $z < 0$ is inside the solid object, $z = 0$ is on the object’s surface, and $z > 0$ is outside. Using the Heaviside step function $H$, the nearby density is formally defined as $\rho = H(-z) \rho_0 + H(z) \rho_1$, the external forces term is defined as $\vec{F} = H(-z) \vec{F}_0 + H(z) (\vec{g} + \vec{e})$, and the acceleration is $\vec{a} = H(-z) \vec{a}_0 + H(z) (\vec{g} + \vec{e})$. Since we solve for vorticity, we can replace $\vec{F}$ with any $\vec{F} + \vec{G}$ to satisfy the boundary condition as long as $\nabla \times \vec{G} = 0$ in $\Omega^c$. First let us consider the following $\vec{G}$

$$\vec{G} = H(z) \frac{\nabla \log(\rho)}{\frac{dz}{dt}} - \frac{\vec{g}}{2} - \frac{\vec{e}}{2} \hspace{1cm} (49)$$

Although $\frac{\nabla \log(\rho)}{\frac{dz}{dt}}$ may have curl, we discretize the object surface in regions of constant $\frac{dz}{dt}$. To prevent the space inside objects from contributing to the non-rigid fluid motion, we set $\rho_0 \to \infty$, thus enforcing $\vec{F}$ inside objects. We take the limit of the integral of the collision term near the surface using the following limit to the heaviside step function $H(z) = \lim_{\rho_0 \to \infty} \frac{1}{2 + \frac{\rho_0}{\tan^{-1}(\frac{z}{\rho_0})}}$, and obtain the surface vorticity

$$\begin{align*}
&\lim_{\rho_0 \to \infty} \lim_{h \to 0} \int_{\rho_0 - h}^{\rho_0} \nabla \log(\rho) \times (\vec{F} + \vec{G} - \frac{\vec{g}}{2} + \frac{\vec{e}}{2}) \\
&= (\vec{F} - \vec{a} - \vec{a}) \times \vec{n}
\end{align*}$$  \hspace{1cm} (50)

References


