Implicit Methods

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PIXAR
ANIMATION STUDIOS

“Give me Stability
or
Give me Death”

— Baraff’s other motto

stability is all stability is all stability is all

- If your step size is too big, your simulation blows up. It isn't pretty.
- Sometimes you have to make the step size so small that you never get anyplace.
- Nasty cases: cloth, constrained systems.

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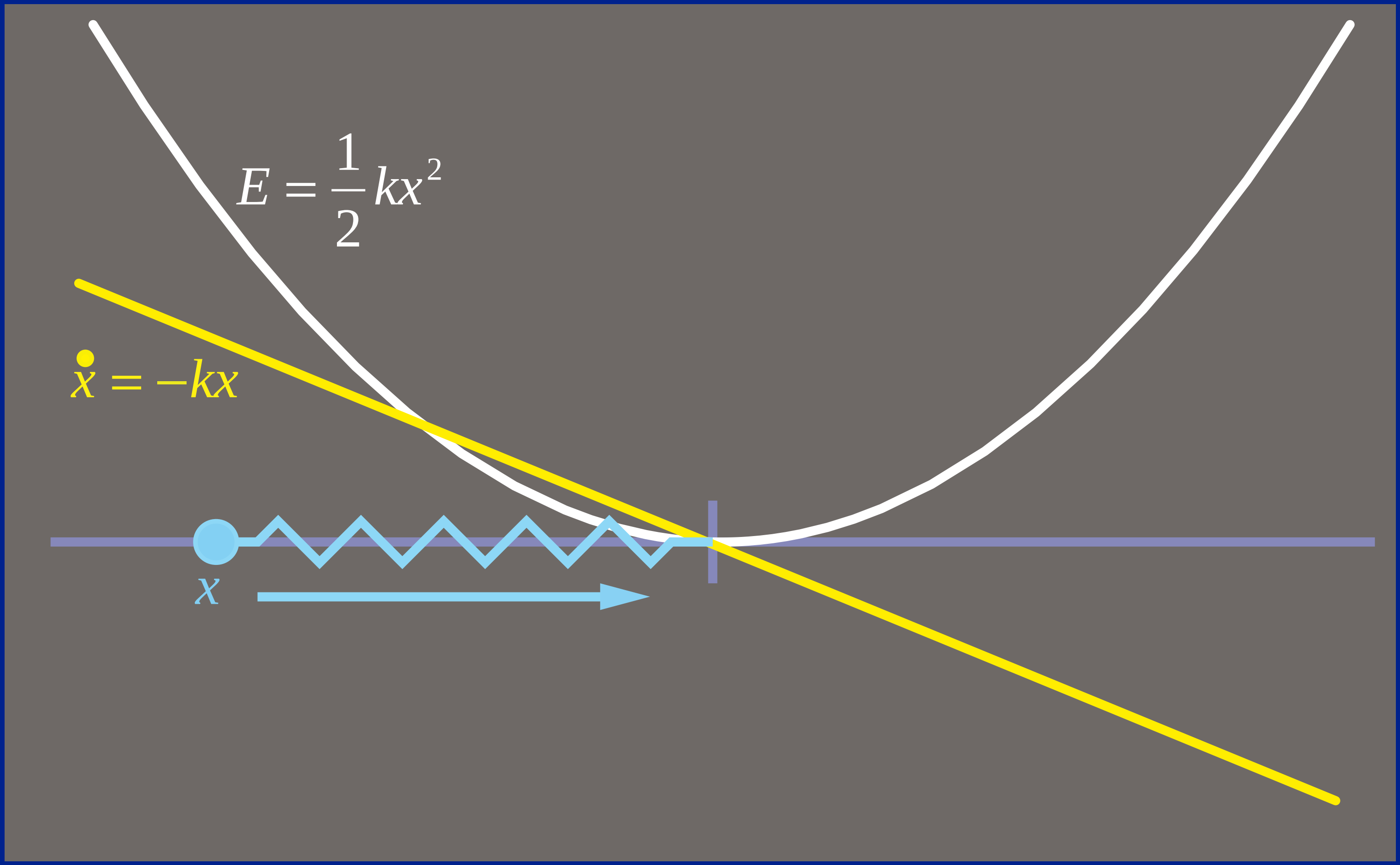
- If your step size is too big, your simulation blows up. It isn't pretty.
- Sometimes you have to make the step size so small that you never get anyplace.
- Nasty cases: cloth, constrained systems.
- Solutions:
  - –Now: use explosion-resistant methods.
  - –Later: reformulate the problem.

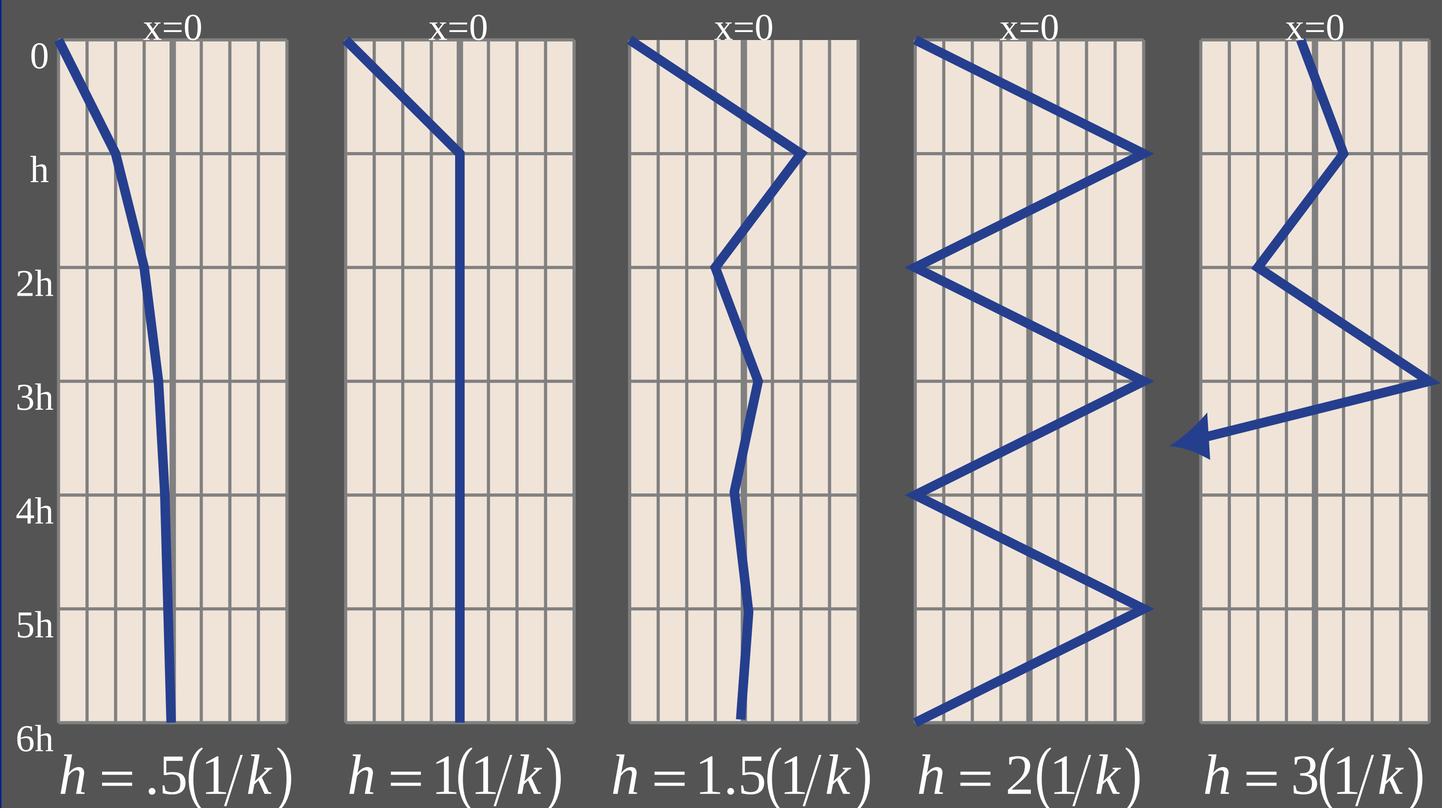
A very simple equation

A 1-D particle governed by $\dot{x} = -kx$ where k is a stiffness constant.



Euler's method has a speed limit

$$x = -kx \qquad \Delta x = -hkx$$

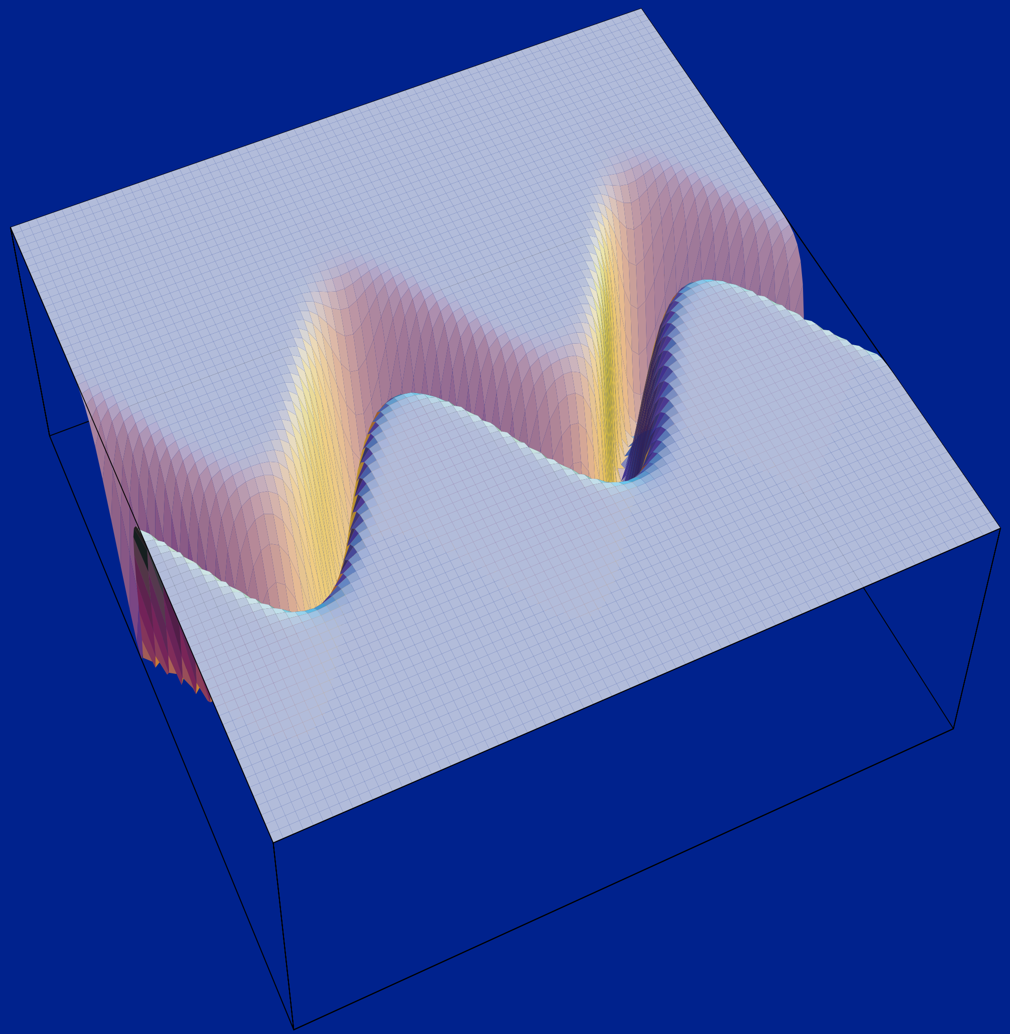


$h > 1/\mathrm{k}$: oscillate. $h > 2/k$: explode!

Stiff Equations

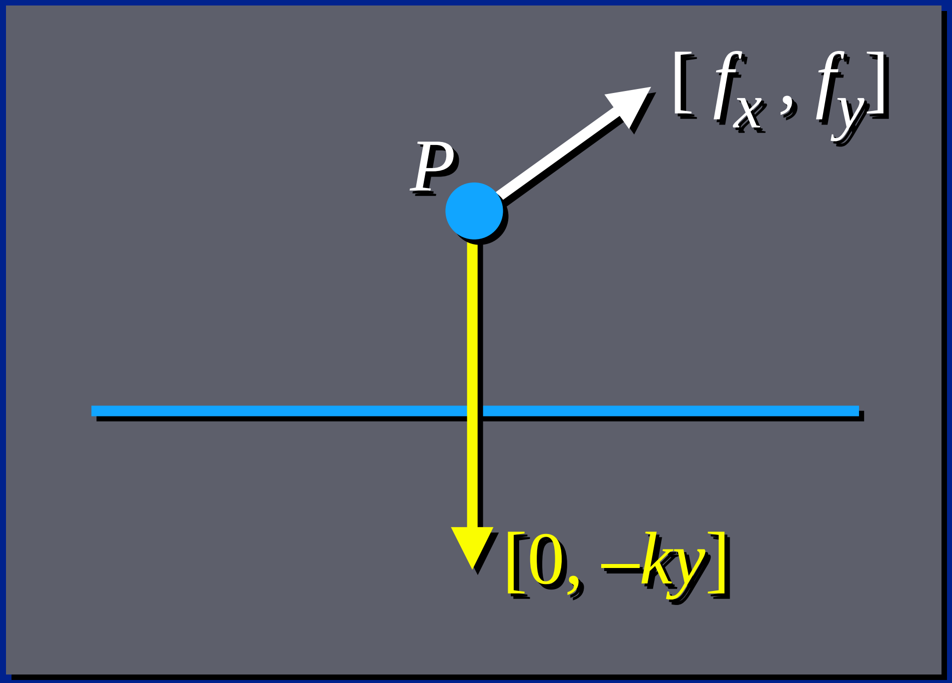
- In more complex systems, step size is limited by the largest *k*. One stiff spring can screw it up for everyone else.
- Systems that have some big *k*'s mixed in are called stiff systems.

A Stiff Energy Landscape



Example: particle-on-line

- A particle *P* in the plane.
- Interactive "dragging" force $[f_x, f_y]$.
- A penalty force $[0,-ky]$ tries to keep *P* on the *x*-axis.



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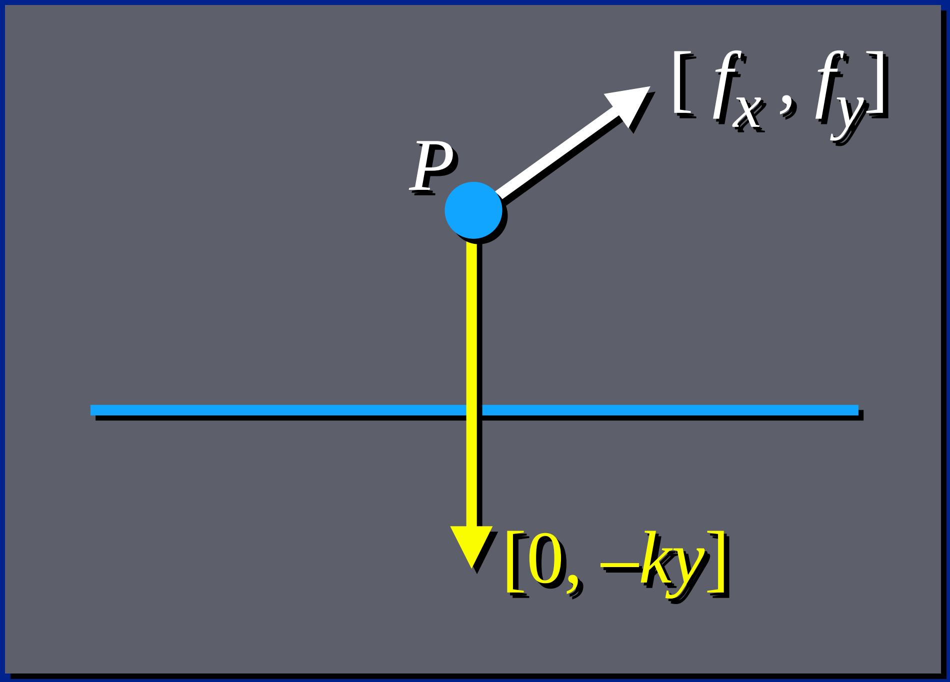
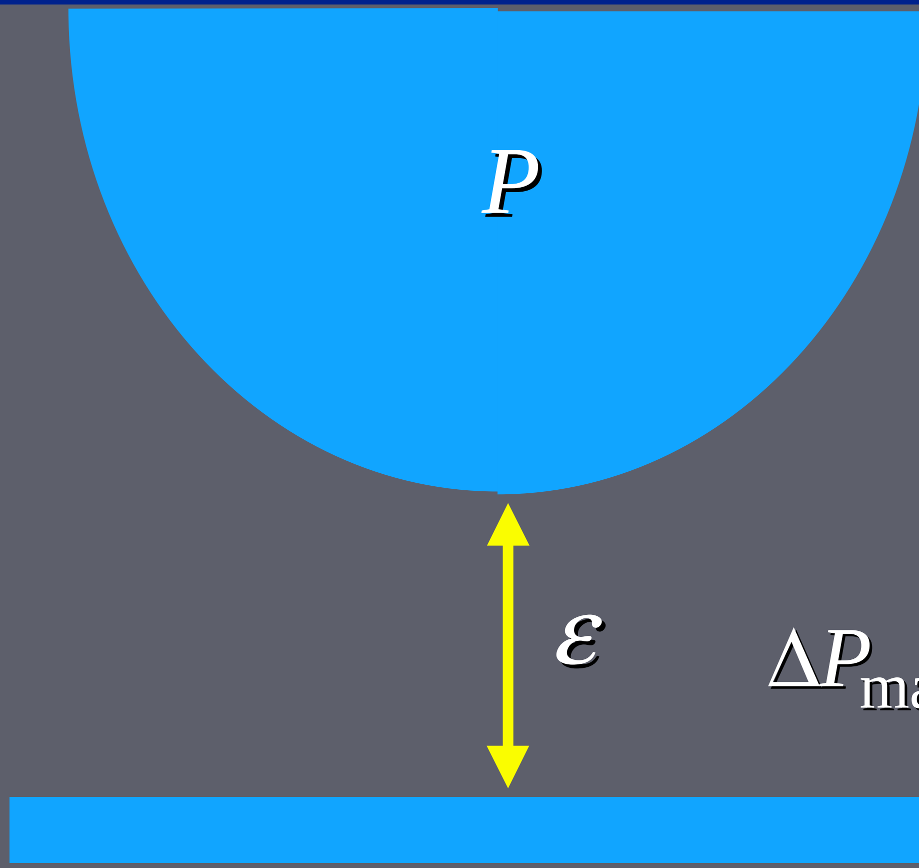


- Suppose you want P to stay within a miniscule ε of the x-axis when you try to pull it off with a huge force f_{max}.
- How big does k have to be? How *small* must h be?

Really big k. Really small h.



$$k = \frac{f_{\max}}{\varepsilon}$$

$$h < \frac{\varepsilon}{f_{\max}}$$

$$\Delta P_{\max} = h f_{\max} < \varepsilon$$

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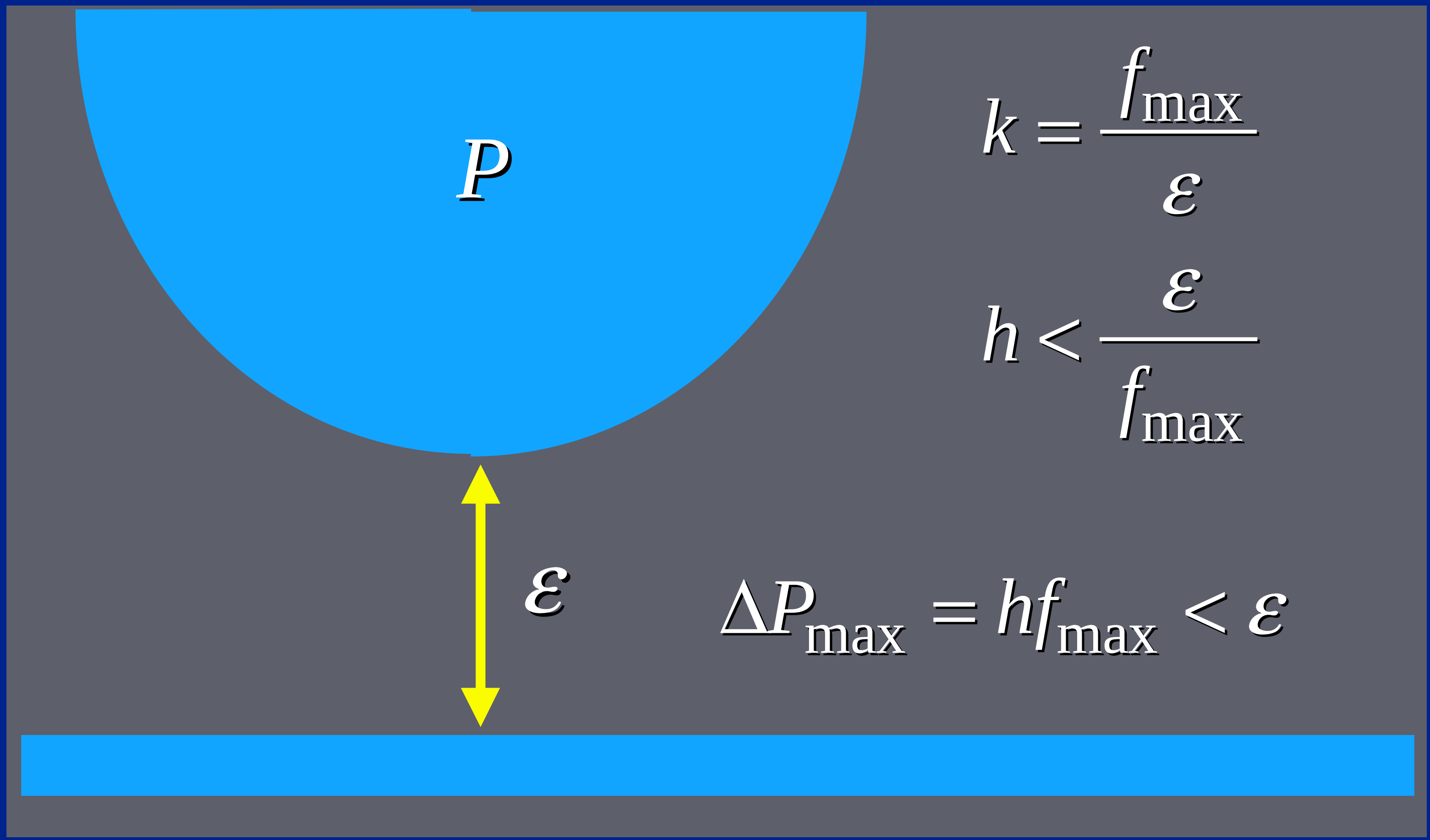


$$k = \frac{f_{\max}}{\varepsilon}$$

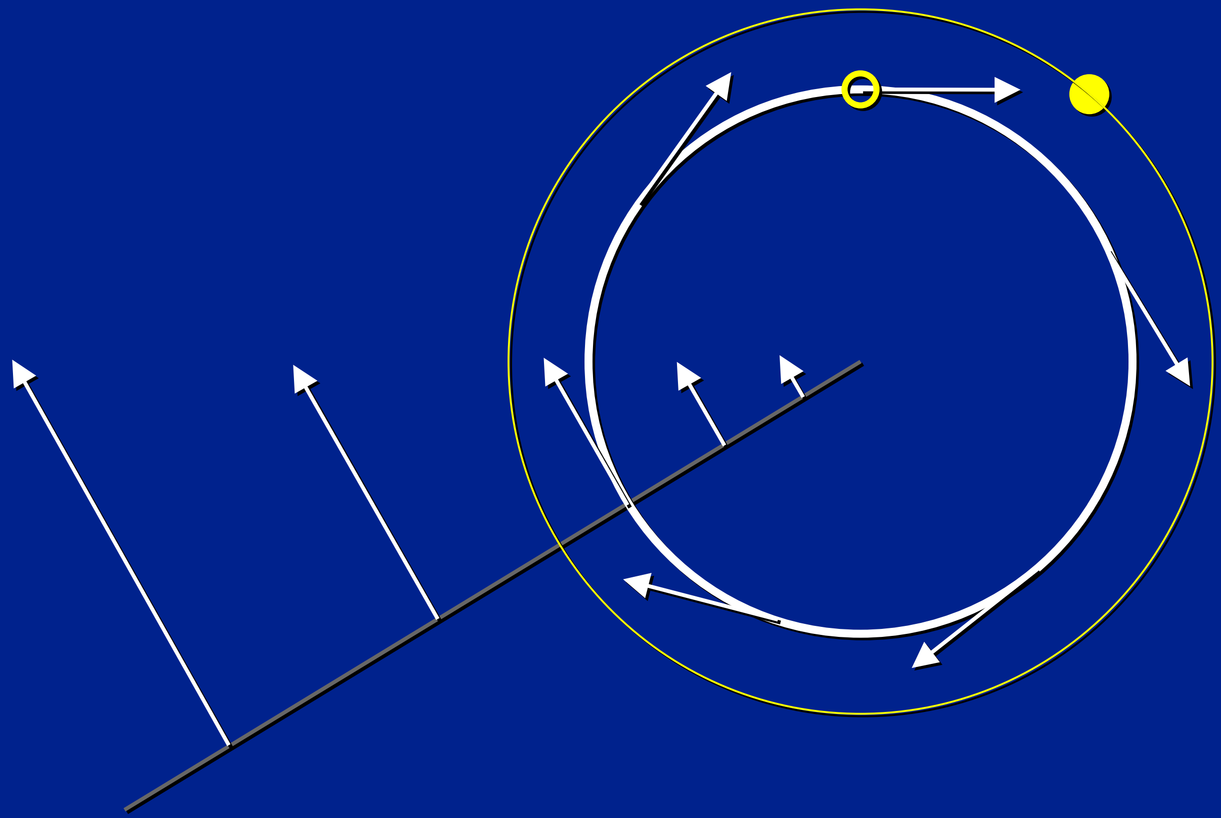
$$h < \frac{\varepsilon}{f_{\max}}$$

$$\Delta P_{\max} = h f_{\max} < \varepsilon$$

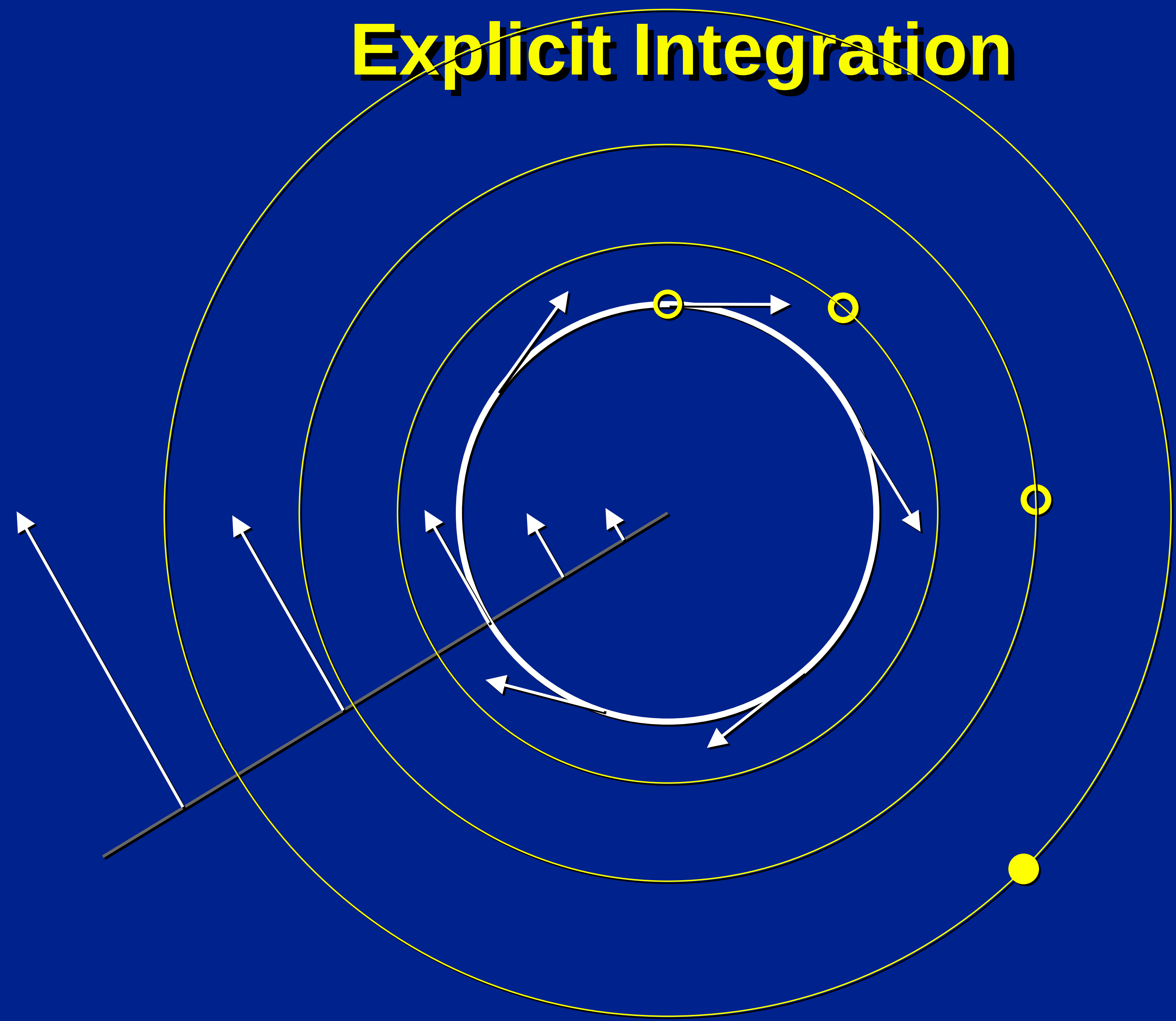
Answer: *h* has to be so small that *P* will never move more than ε per step.

Result: Your simulation grinds to a halt.

Explicit Integration



Explicit Integration



(Explicit) Euler Method

$$x(t_0 + h) = x(t_0) + h\,\dot{x}(t_0)$$

Implicit Euler Method

$$x(t_0 + h) = x(t_0) + h\,\dot{x}(t_0)$$

$$x(t_0 + h) = x(t_0) + h\,\dot{x}(t_0 + \Delta t)$$

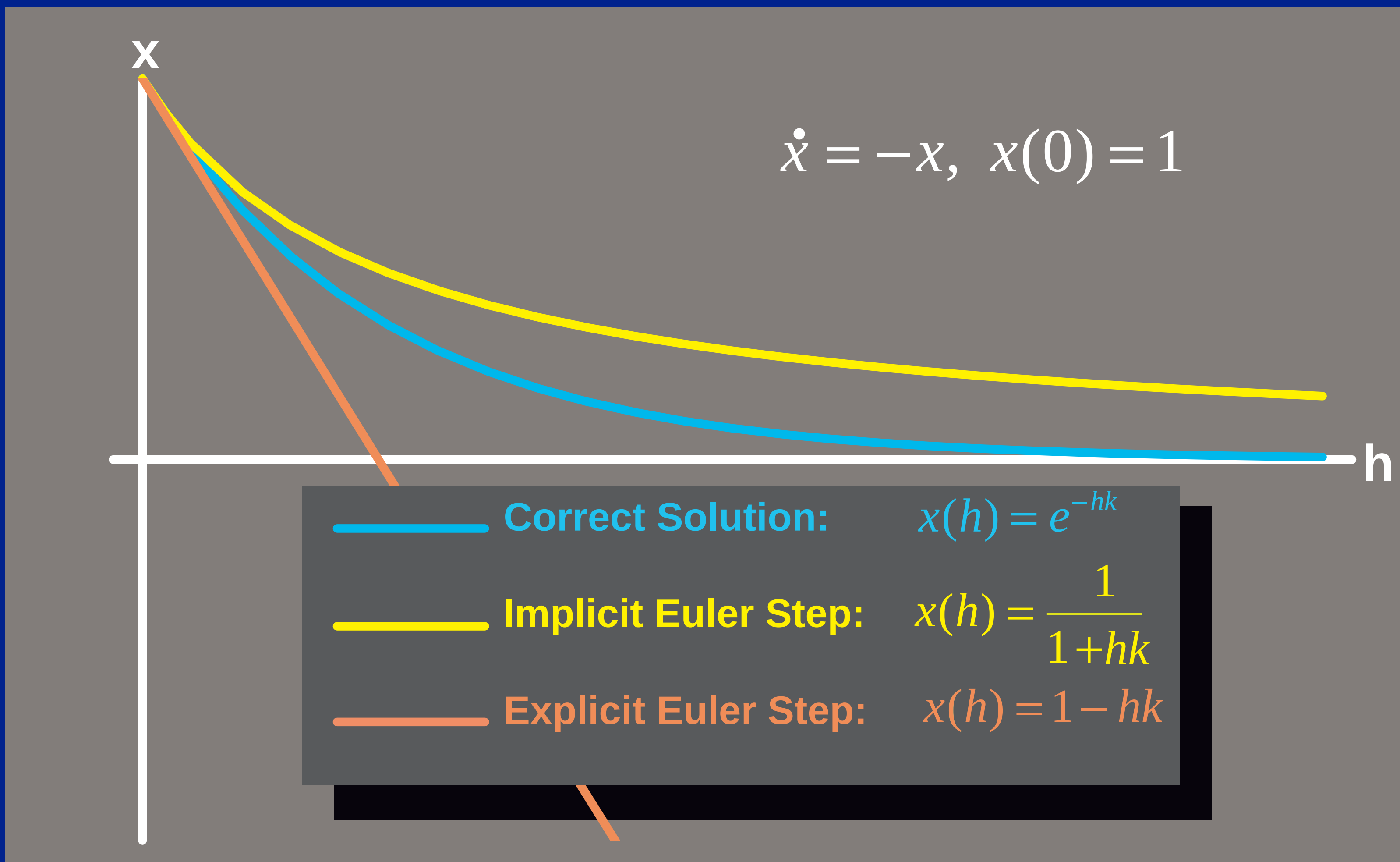
Implicit Euler for $\dot{x} = -kx$

$$x(t+h) = x(t) + h\,\dot{x}(t+h)$$

$$= x(t) - h\,kx(t+h)$$

$$= \frac{x(t)}{1+hk}$$

One Step: Implicit vs. Explicit

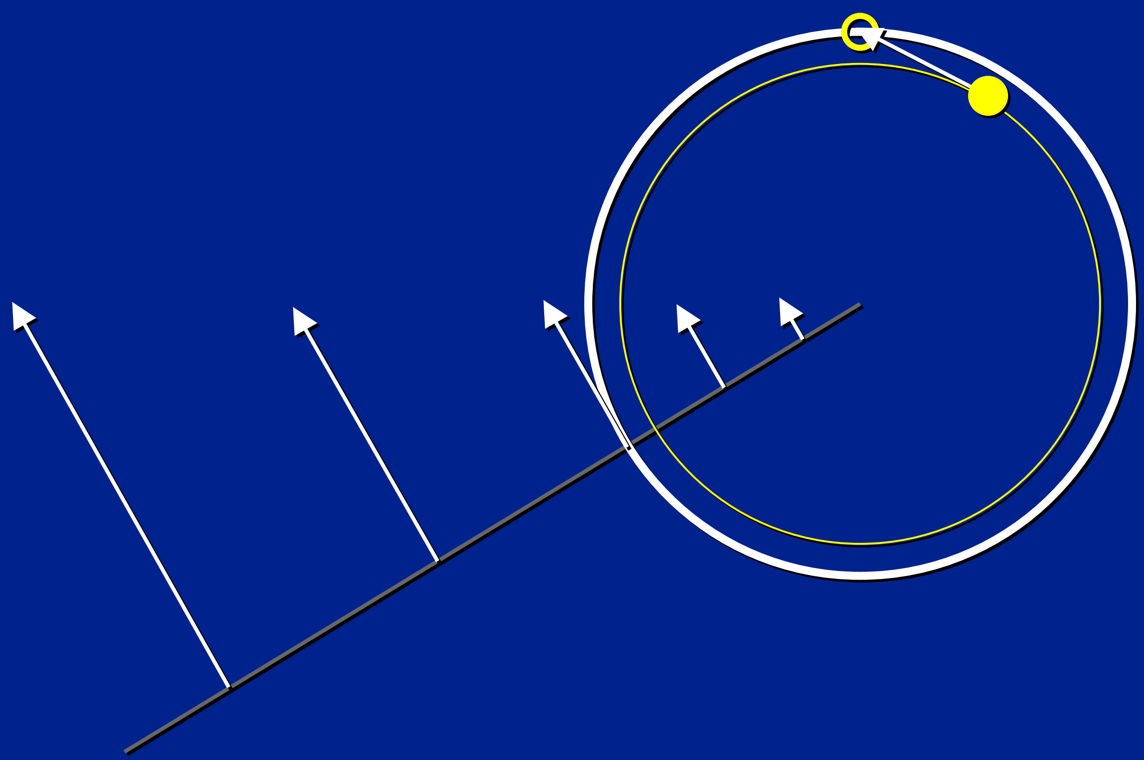


Large Systems

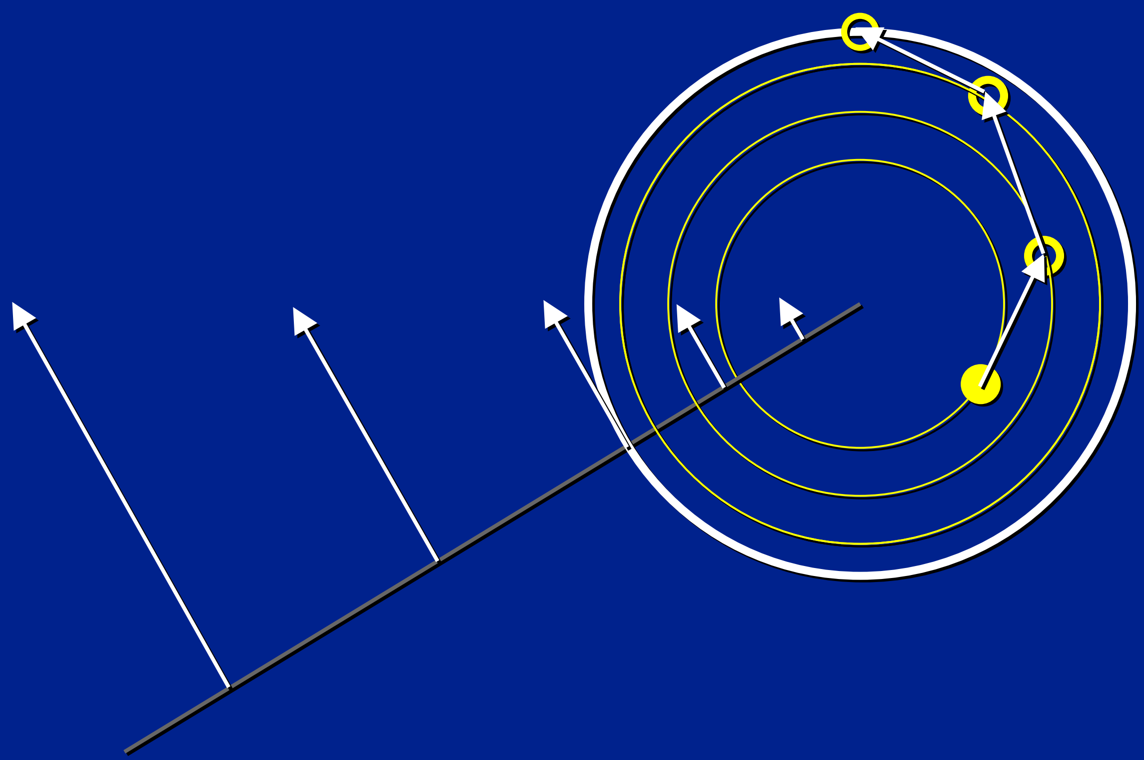
$$\frac{d}{dt}\mathbf{X}(t) = \dot{\mathbf{X}}(t) = f\left(\mathbf{X}(t)\right)$$

$$\Delta\mathbf{X}(t_0) = h\,\dot{\mathbf{X}}(t_0 + \Delta t) = h\,f\left(\mathbf{X}(t_0 + \Delta t)\right)$$
$$= h\,f\left(\mathbf{X}(t_0) + \Delta\mathbf{X}(t_0)\right)$$

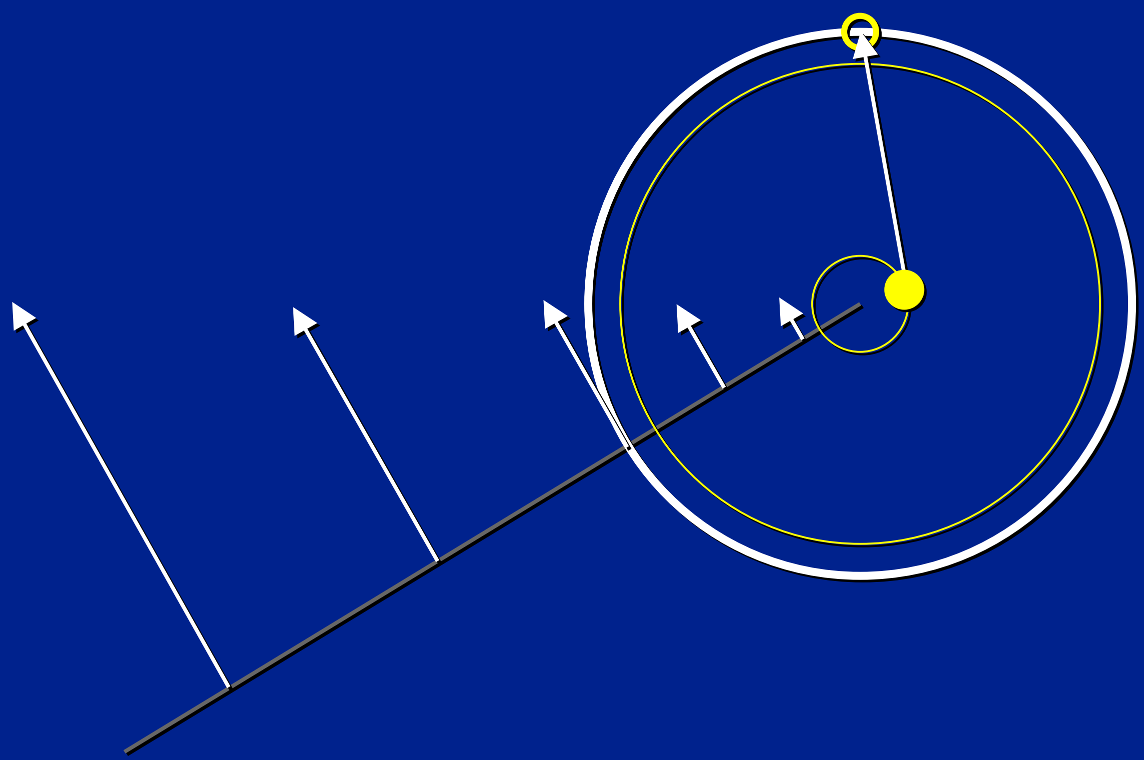
Implicit Integration



Implicit Integration



Implicit Integration (Big Step)



(Linearized) Implicit Integration

$$\dot{\mathbf{X}}(t) = f\left(\mathbf{X}(t)\right)$$

$$\Delta\mathbf{X} = h\, f\left(\mathbf{X}_0 + \Delta\mathbf{X}\right)$$

$$\Delta\mathbf{X} = h\left(f\left(\mathbf{X}_0\right) + \left(\frac{\partial f}{\partial \mathbf{X}}\right)\Delta\mathbf{X} \right)$$

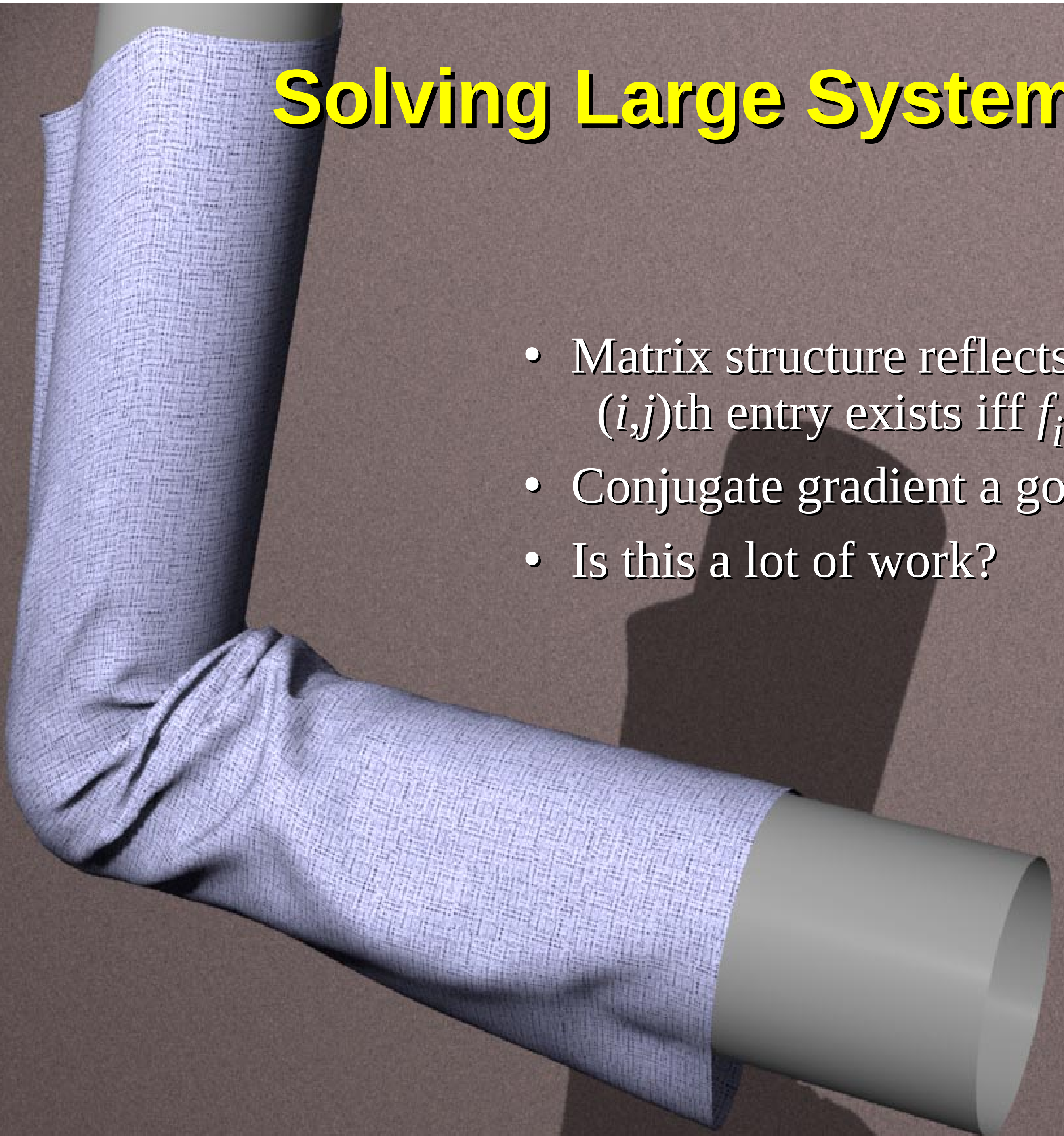
Single-Step Implicit Euler Method

$$\Delta \mathbf{X} = h\left(f(\mathbf{X}_0) + \left(\frac{\partial f}{\partial \mathbf{X}}\right)\Delta \mathbf{X} \right)$$

$$\underbrace{\left(\mathbf{I} - h\frac{\partial}{\partial \mathbf{X}}\left(\dot{\mathbf{X}}(t_0) \right) \right)}_{n \times n \text{ sparse matrix}} \Delta \mathbf{X} = h\,\dot{\mathbf{X}}(t_0)$$

$n \times n$ sparse matrix

Solving Large Systems



- Matrix structure reflects force-coupling: (i,j)th entry exists iff f_i depends on $\mathbf{X}_j$
- Conjugate gradient a good first choice
- Is this a lot of work?